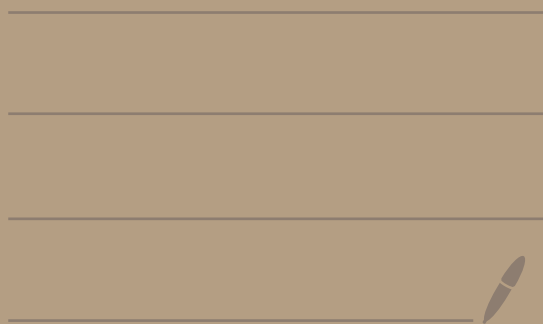


Math 2150-01

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## Topic 10 - Reduction of order

Suppose you know one solution  $y_1$  to the homogeneous ODE

$$y'' + a_1(x)y' + a_0(x)y = 0 \quad (*)$$

on an interval  $I$  where  $y_1(x) \neq 0$  on  $I$ . Then one can find another solution using

$$y_2 = y_1 \int \frac{e^{-\int a_1(x) dx}}{y_1^2} dx$$

Further,  $y_1$  and  $y_2$  will be linearly independent. Thus,

$y_h = c_1 y_1 + c_2 y_2$   
will give all solutions to  $(*)$ .

[Derivation is in online notes]

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Ex: Consider

$$(x^2+1)y'' - 2xy' + 2y = 0 \quad (**)$$

on  $I = (0, \infty)$ .

Let  $y_1 = x$ .

Then  $y_1 = x$  solves  $(**)$  since if you plug it in you get

$$(x^2+1) \cdot 0 - 2x(1) + 2(x) = 0$$

Let's find our second solution.

First we must put a 1 in front of  $y''$  in  $(**)$ .

Divide by  $(x^2+1)$  to get

$$y'' - \underbrace{\frac{2x}{x^2+1}}_{a_1(x)} y' + \frac{2}{x^2+1} y = 0$$

We get

$$y_2 = y_1 \int \frac{e^{-\int a_1(x) dx}}{y_1^2} dx$$

$$= x \int \frac{e^{-\int \frac{-2x}{x^2+1} dx}}{(x)^2} dx$$

$$= x \int \frac{e^{\int \frac{2x}{x^2+1} dx}}{x^2} dx$$

$$\int \frac{2x}{x^2+1} dx = \int \frac{1}{u} du = \ln|u|$$

$$= \ln|x^2+1|$$

$$= \ln(x^2+1)$$

$u = x^2+1$   
 $du = 2x dx$

$x^2+1 > 0$

$$= x \int \frac{e^{\ln(x^2+1)}}{x^2} dx$$

$$= x \int \frac{x^2+1}{x^2} dx$$



$$e^{\ln(A)} = A$$

$$= x \int \left( \frac{x^2}{x^2} + \frac{1}{x^2} \right) dx$$

$$= x \int (1 + x^{-2}) dx$$

$$= x \left( x + \frac{x^{-1}}{-1} \right)$$

$$= x \left( x - \frac{1}{x} \right)$$

$$= x^2 - 1$$

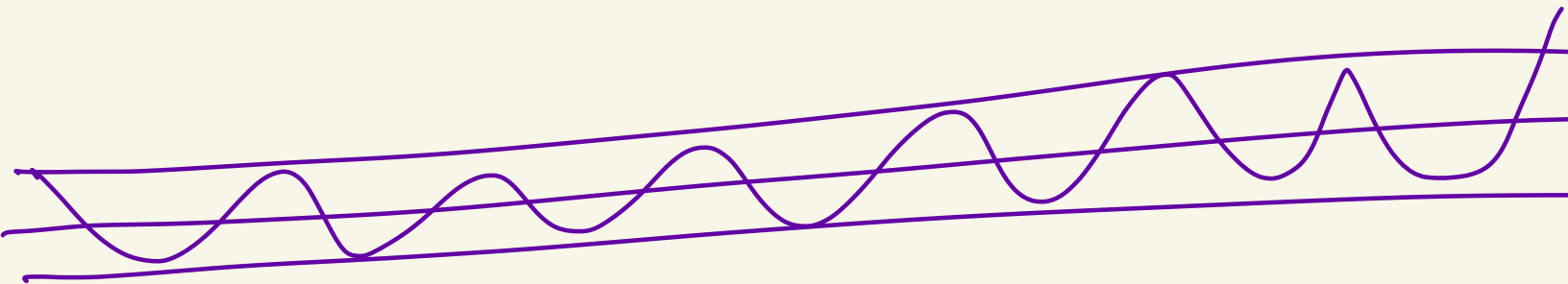
$$\text{So, } y_1 = x, y_2 = x^2 - 1.$$

Thus, the general solution to

$$(x^2 + 1)y'' - 2xy' + 2y = 0$$

is

$$\begin{aligned} y_h &= c_1 y_1 + c_2 y_2 \\ &= c_1 x + c_2 (x^2 - 1) \end{aligned}$$



## HW 10 - #1(a)

Given that  $y_1 = x^4$  is a solution to

$$x^2 y'' - 7xy' + 16y = 0$$

on  $I = (0, \infty)$ ,

find the general solution.

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First divide by  $x^2$  to put a 1 in front of  $y''$ .

We get

$$y'' - \frac{7}{x} y' + \frac{16}{x^2} y = 0$$

$$a_1(x) = -\frac{7}{x}$$

We get

$$y_2 = y_1 \int \frac{e^{-\int a_1(x) dx}}{y_1^2} dx$$

$$= x^4 \int \frac{e^{-\int -\frac{7}{x} dx}}{(x^4)^2} dx$$

$$= x^4 \int \frac{e^{\int \frac{7}{x} dx}}{x^8} dx$$

$$= x^4 \int \frac{e^{7 \int \frac{1}{x} dx}}{x^8} dx$$

$$= x^4 \int \frac{e^{7 \ln(x)}}{x^8} dx$$

$\int \frac{1}{x} dx = \ln|x| = \ln(x) \quad \begin{matrix} x > 0 \\ \uparrow \\ I = (0, \infty) \end{matrix}$



$$= x^4 \int \frac{e^{\ln(x^7)}}{x^8} dx$$

$$A \ln(B) = \ln(B^A)$$

$$= x^4 \int \frac{x^7}{x^8}$$

$$= x^4 \int \frac{1}{x} dx$$

$$= x^4 \ln|x|$$

$$= x^4 \ln(x)$$

$$I = (0, \infty) \\ x > 0$$

Thus the general solution to

$$x^2 y'' - 7xy' + 16y = 0$$

on  $I = (0, \infty)$  is

$$y_h = c_1 y_1 + c_2 y_2$$
$$= c_1 x^4 + c_2 x^4 \ln(x)$$