Math 2150-01
3/26/25

Topic 10 - Reduction of order

Suppose you know one solution y_1 to the homogeneous ODE

$$\boxed{y'' + a_1(x)y' + a_0(x)y = 0} \quad (*)$$

on an interval I where $y_1(x) \neq 0$ on I. Then one can find another solution using

$$\boxed{y_2 = y_1 \int \frac{e^{-\int a_1(x)\,dx}}{y_1^2}\,dx}$$

Further, y_1 and y_2 will be linearly independent. Thus,

$$y_h = c_1 y_1 + c_2 y_2$$

will give all solutions to $(*)$.

[Derivation is in online notes]

Ex: Consider

$$(x^2+1)y'' - 2xy' + 2y = 0 \qquad (**)$$

on $I = (0, \infty)$.

Let $y_1 = x$.

Then $y_1 = x$ solves $(**)$ since if you plug it in you get

$$(x^2+1)\cdot 0 - 2x(1) + 2(x) = 0$$

Let's find our second solution.

First we must put a 1 in front of y'' in $(**)$.

Divide by (x^2+1) to get

$$y'' \underbrace{- \frac{2x}{x^2+1}}_{a_1(x)} y' + \frac{2}{x^2+1} y = 0$$

We get

$$y_2 = y_1 \int \frac{e^{-\int a_1(x)dx}}{y_1^2} dx$$

$$= x \int \frac{e^{-\int -\frac{2x}{x^2+1}dx}}{(x)^2} dx$$

$$= x \int \frac{e^{\int \frac{2x}{x^2+1}dx}}{x^2} dx$$

$$\int \frac{2x}{x^2+1} dx = \int \frac{1}{u} du = \ln|u|$$

$$= \ln|x^2+1|$$

$$= \ln(x^2+1)$$

$u = x^2+1$

$du = 2x\,dx$

$x^2+1>0$

$$= x \int \frac{e^{\ln(x^2+1)}}{x^2}\,dx$$

$$= x \int \frac{x^2+1}{x^2}\,dx$$

$$e^{\ln(A)} = A$$

$$= x \int \left(\frac{x^2}{x^2} + \frac{1}{x^2}\right) dx$$

$$= x \int (1 + x^{-2})\,dx$$

$$= x\left(x + \frac{x^{-1}}{-1}\right)$$

$$= x\left(x - \frac{1}{x}\right)$$

$$= x^2 - 1$$

So, $y_1 = x$, $y_2 = x^2 - 1$.

Thus, the general solution to

$$(x^2+1)y'' - 2xy' + 2y = 0$$

is

$$\begin{aligned} y_h &= c_1 y_1 + c_2 y_2 \\ &= c_1 x + c_2 (x^2 - 1) \end{aligned}$$

HW 10 - #1(a)

Given that $y_1 = x^4$ is a solution to

$$x^2y'' - 7xy' + 16y = 0$$

on $I = (0, \infty)$, find the general solution.

First divide by x^2 to put a 1 in front of y''.

We get

$$y'' \underbrace{- \frac{7}{x}}\, y' + \frac{16}{x^2} y = 0$$

$$a_1(x) = -\frac{7}{x}$$

We get

$$y_2 = y_1 \int \frac{e^{-\int a_1(x)\,dx}}{y_1^2}\,dx$$

$$= x^4 \int \frac{e^{-\int -\frac{7}{x}dx}}{(x^4)^2}\,dx$$

$$= x^4 \int \frac{e^{\int \frac{7}{x}dx}}{x^8}\,dx$$

$$= x^4 \int \frac{e^{7\int \frac{1}{x}dx}}{x^8}\,dx$$

$$= x^4 \int \frac{e^{7\ln(x)}}{x^8}\,dx$$

$$\boxed{\int \frac{1}{x}dx = \ln|x| = \ln(x) \quad x>0}$$

$$I = (0, \infty)$$

$$= x^4 \int \frac{e^{\ln(x^7)}}{x^8}\,dx$$

$A\ln(B)$
$= \ln(B^A)$

$$= x^4 \int \frac{x^7}{x^8}$$

$$= x^4 \int \frac{1}{x}\,dx$$

$$= x^4 \ln|x|$$

$I=(0,\infty)$
$x>0$

$$= x^4 \ln(x)$$

Thus the general solution to

$$x^2y'' - 7xy' + 16y = 0$$

on $I=(0,\infty)$ is

$$y_h = c_1 y_1 + c_2 y_2$$
$$= c_1 x^4 + c_2 x^4 \ln(x)$$