Math 2150-01
4/14/25

Schedule

4/14 TOPIC 12	4/16 TOPIC 12
4/21 REVIEW	4/23 REVIEW
4/28 TEST 2	4/30 TOPIC 13
5/5 TOPIC 13 (HAND BACK TESTS)	5/7 REVIEW FOR FINAL
5/12	5/14 FINAL 2:30-4:30

HW 11 (1)(g)

Find the power series for $f(x) = \frac{1}{x^2}$ centered at $x_0 = 1$

Use

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \dots$$

$x_0 = 1$

$f(x) = \frac{1}{x^2} = x^{-2}$ ← $\boxed{f(1) = 1}$

$f'(x) = -2x^{-3} = \frac{-2}{x^3}$ ← $\boxed{f'(1) = -2}$

$$f''(x) = 6x^{-4} = \frac{6}{x^4} \leftarrow \boxed{f''(1) = 6}$$

$$f'''(x) = -24x^{-5} = \frac{-24}{x^5} \leftarrow \boxed{f'''(1) = -24}$$

Let's stop here.

$$\underbrace{f(x)}_{1/x^2} = 1 - 2(x-1) + \frac{6}{2!}(x-1)^2 - \frac{24}{3!}(x-1)^3 + \cdots$$

So,

$$\boxed{\frac{1}{x^2} = 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + \cdots}$$

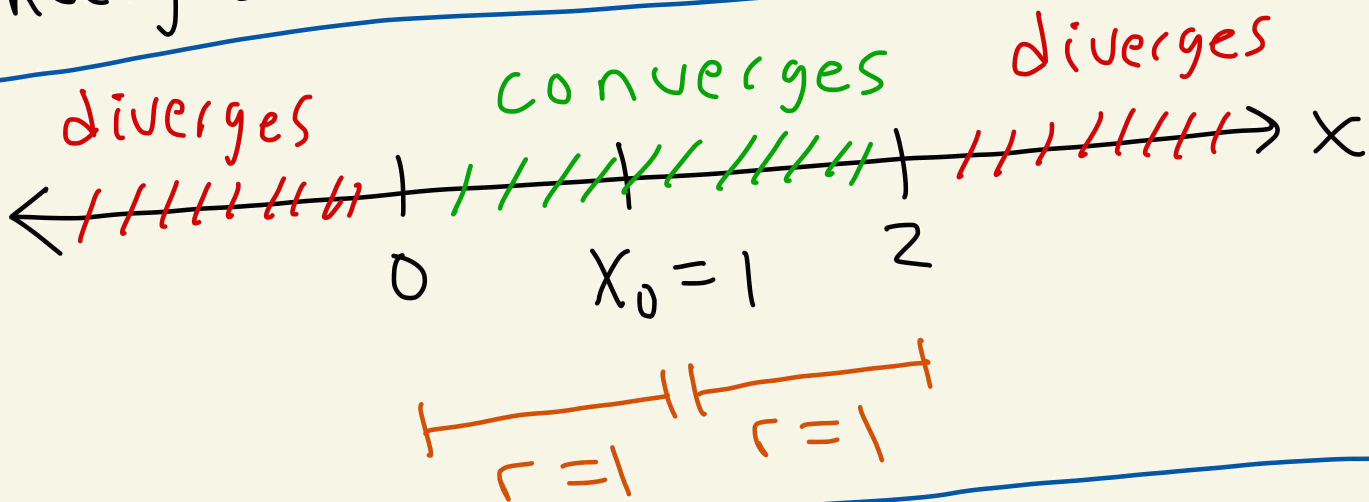
But we do not know the radius of convergence.

Let's see how we could have found that.

Last time we saw that

$$\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \dots$$

converges when $0 < x < 2$



radius of convergence is $r = 1$.

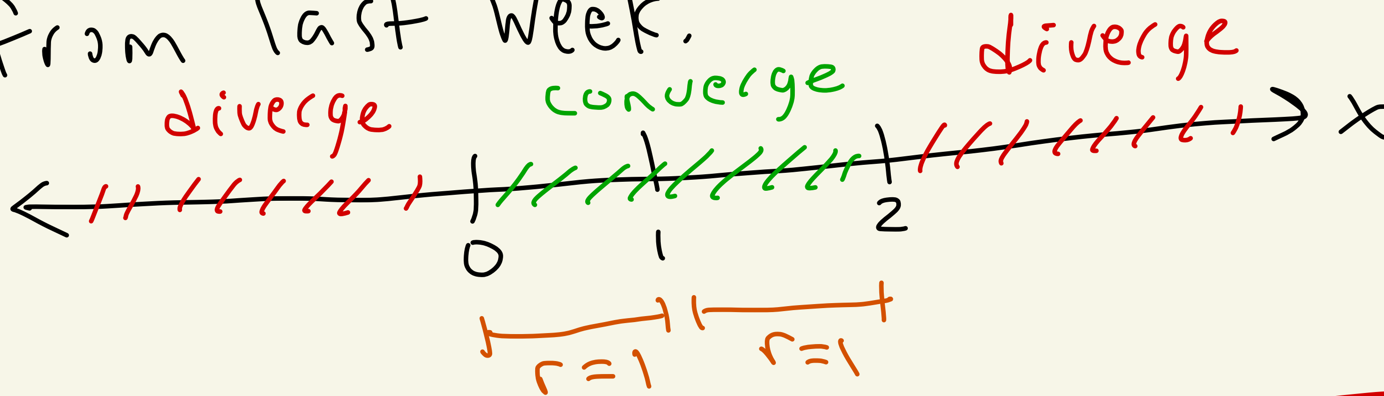
Differentiate both sides to get

$$-\frac{1}{x^2} = 0 - 1 + 2(x-1) - 3(x-1)^2 + \dots$$

So,

$$\frac{1}{x^2} = 1 - 2(x-1) + 3(x-1)^2 - \dots$$

will have same radius of convergence $r=1$ by theorem from last week.

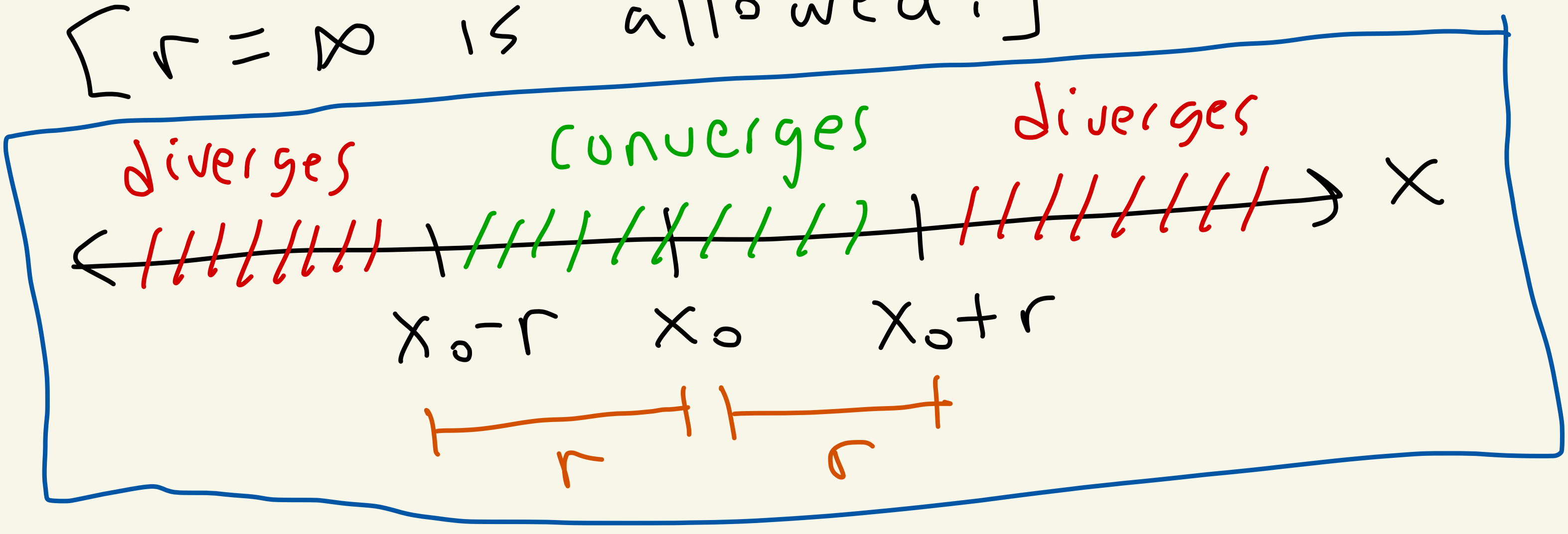


Topic 12 - Power series solutions to ODEs

Def: We say that a function $f(x)$ is analytic at x_0 if it has a power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$$

centered at x_0 with positive radius of convergence $r > 0$.
[$r = \infty$ is allowed.]



Ex: $x_0 = 0$

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$$

has radius of convergence $r = \infty$

So, $\sin(x)$ is analytic at $x_0 = 0$

Ex: $x_0 = 1$

$$\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \dots$$

has radius of convergence $r = 1$.

Thus, $\frac{1}{x}$ is analytic at $x_0 = 1$.

Ex: $x_0 = 2$

$$x^2 = 4 + 4(x-2) + (x-2)^2$$ last week

has radius of convergence $r = \infty$

Thus, x^2 is analytic at $x_0 = 2$

Facts:

- polynomials are analytic for all x_0
- e^x, $\sin(x)$, $\cos(x)$ are analytic for all x_0
- rational functions (ratio of polynomials) are analytic at all x_0 except possibly where the denominator is zero.

Ex: $x^2 - 5x + 2$

is analytic for all x_0.

It's a polynomial.

Ex: $\dfrac{x}{x^2 - 1}$ ← rational function

is analytic for all x_0

except $\underbrace{x_0 = 1, -1}_{\text{when } x^2 - 1 = 0}$

Main Theorem

Consider either of the initial value problems:

$$\boxed{\begin{gathered} y' + a_0(x)\,y = b(x) \\ y(x_0) = y_0 \end{gathered}}$$

first order

OR

$$\boxed{\begin{gathered} y'' + a_1(x)\,y' + a_0(x)\,y = b(x) \\ y'(x_0) = y_0',\ y(x_0) = y_0 \end{gathered}}$$

second order

In either case, if the $a_i(x)$ and $b(x)$ are analytic at x_0 then there exists a unique solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

centered at x_0.

Furthermore, the radius of convergence $r > 0$ for the power series of the solution y is at least the smallest radius of convergence from amongst the power series of the $a_i(x)$ and $b(x)$.

<u>Ex:</u>

$$y' + \overbrace{a_1(x)}^{\text{radius of convergence } r=2 \text{ at } x_0} y = \overbrace{b(x)}^{\text{radius of convergence } r=3 \text{ at } x_0}$$

$$y(x_0) = y_0$$

Then the solution will have radius of convergence at least $r = 2$.