Math 2150-01
4/23/25

HW 11

① (f) Find a power series for $f(x) = \frac{1}{x}$ centered at $x_0 = 1$. What's its radius of convergence.

Hint: Use

$$\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \dots$$

with radius of convergence $r = 1$

Differentiate both sides to get

$$\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots$$

with radius of convergence $r = 1$

HW 12
3

Find the first four non-zero terms of a power series solution to

$$\boxed{\begin{gathered} xy'' + x^2y' - 2y = 0 \\ y'(1) = 1, \quad y(1) = 1 \end{gathered}} \leftarrow x_0 = 1$$

Find the radius of convergence r of the answer.

What will r be for the solution we find?

Divide by x to get

$$y'' + \underbrace{x}\, y' - \underbrace{\frac{2}{x}}\, y = \underbrace{0}$$

Coefficients

$x \leftarrow$ polynomial, $r = \infty$

$-\frac{2}{x} = -2\left(\frac{1}{x}\right) \leftarrow$ just did above, $r = 1$

$0 \leftarrow$ polynomial, $r = \infty$

Our answer will have radius at least $r = 1$. (min of $\infty, 1, \infty$)

Let's find the solution.

Given: $\boxed{\begin{array}{l} y(1) = 1 \\ y'(1) = 1 \end{array}}$

Have:

$$y'' = -xy' + \frac{2}{x} y$$

$$y''(1) = -(1)\left[y'(1)\right] + \frac{2}{1}\left[y(1)\right]$$

$$= -1 \cdot [1] + 2[1] = 1$$

$$y''(1) = 1$$

Differentiate $y'' = -xy' + 2x^{-1}y$ to get

$$y''' = (-1)y' - xy'' - 2x^{-2}y + 2x^{-1}y'$$

$$y'''(1) = (-1)[y'(1)] - (1)[y''(1)] - 2(1)^{-2}[y(1)] + 2(1)^{-1}[y'(1)]$$

$$y'''(1) = (-1)[1] - (1)[1] - 2(1)[1] + 2(1)[1]$$

$$y'''(1) = -2$$

Plug it into the Taylor series.

$$y(x) = y(1) + y'(1)(x-1) + \frac{y''(1)}{2!}(x-1)^2$$

$$+ \frac{y'''(1)}{3!}(x-1)^3 + \ldots$$

$$\boxed{\begin{aligned} &\text{So,} \\ &y(x) = 1 + 1\cdot(x-1) + \frac{1}{2!}(x-1)^2 \\ &\qquad + \frac{-2}{3!}(x-1)^3 + \ldots \\ &\quad = 1 + (x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{3}(x-1)^3 + \ldots \\ &\text{with } r = 1 \end{aligned}}$$

HW 10

1(e) Given that $y_1 = x^{-4}$ is a solution to

$$x^2 y'' - 20y = 0$$

on $I = (0, \infty)$, find another linearly independent solution. Then state the general solution

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Divide by x^2 to get

$$\underbrace{y'' - \frac{20}{x^2} y}_{} = 0$$

Coefficient $a_1(x)$ of y' is $a_1(x) = 0$

Then,

$$y_2 = y_1 \int \frac{e^{-\int a_1(x)\,dx}}{y_1^2}\,dx$$

$$= x^{-4} \int \frac{e^{-\int 0\,dx}}{(x^{-4})^2}\,dx$$

$$= x^{-4} \int \frac{e^{-0}}{(x^{-4})^2}\,dx$$

$$= x^{-4} \int \frac{1}{x^{-8}}\,dx$$

$$= x^{-4} \int x^{8}\,dx$$

$$= x^{-4}\left(\frac{1}{9}x^{9}\right)$$

$= \frac{1}{9} x^5$

So, $y_2 = \frac{1}{9} x^5$ and the general solution to $x^2 y'' - 20y = 0$ is

$$y_h = c_1 y_1 + c_2 y_2$$
$$= c_1 x^{-4} + c_2 \left(\frac{1}{9} x^5\right)$$

HW 9

1(a) Find the general solution to

$$y'' + y = \sec(x)$$

Use variation of parameters to find y_p

Step 1: Solve the homogeneous eqn

$$y'' + y = 0$$

We have

$$r^2 + 1 = 0$$

$$r = \frac{-0 \pm \sqrt{0^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{\pm\sqrt{-4}}{2} = \frac{\pm\sqrt{4}\sqrt{-1}}{2}$$

$$= \frac{\pm 2i}{2} = \pm i = \underbrace{0 \pm 1 \cdot i}_{\substack{\alpha \pm \beta i \\ \alpha = 0,\ \beta = 1}}$$

$$y_h = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x)$$

$$= c_1 e^0 \cos(x) + c_2 e^0 \sin(x)$$

$$= c_1 \cos(x) + c_2 \sin(x)$$

Step 2: For

$$y'' + y = \boxed{\sec(x)}$$

$b(x)$

We have

$$y_p = v_1 y_1 + v_2 y_2$$

Where

$$y_1 = \cos(x), \quad y_2 = \sin(x)$$

and

$$v_1 = \int \frac{-y_2 b(x)}{W(y_1, y_2)} dx, \quad v_2 = \int \frac{y_1 b(x)}{W(y_1, y_2)} dx$$

We have

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

$$= \cos(x)\cos(x) - (\sin(x))(-\sin(x))$$

$$= \cos^2(x) + \sin^2(x) = 1$$

Then

$$v_1 = \int \frac{-y_2 b(x)}{W(y_1,y_2)}\,dx = \int \frac{-\sin(x)\sec(x)}{1}\,dx$$

$$= \int \frac{-\sin(x)}{\cos(x)}\,dx = \int -\tan(x)\,dx$$

$$= -\ln|\sec(x)|$$

Check:

$$\int \frac{-\sin(x)}{\cos(x)}\,dx = \int \frac{1}{u}\,du = \ln|u| = \ln|\cos(x)|$$

$$\boxed{\begin{aligned} u &= \cos(x) \\ du &= -\sin(x)\,dx \end{aligned}}$$

$$= \ln\left|\frac{1}{\sec(x)}\right|$$

$$= \ln|\sec(x)^{-1}|$$

$$= -\ln|\sec(x)|$$

$$v_2 = \int \frac{y_1 b(x)}{W(y_1,y_2)}\,dx = \int \frac{\cos(x)\sec(x)}{1}\,dx$$

$$= \int \cos(x) \frac{1}{\cos(x)}\, dx = \int 1\, dx$$

$$= x$$

Then,

$$y_p = v_1 y_1 + v_2 y_2$$
$$= -\ln|\sec(x)| \cos(x) + x \sin(x)$$

The general solution to $y'' + y = \sec(x)$ is

$$y = y_h + y_p$$
$$= c_1 \cos(x) + c_2 \sin(x)$$
$$- \ln|\sec(x)| \cos(x) + x \sin(x)$$

HW 8
1(e)

Find y_p for

$$4y'' - 4y' - 3y = \cos(2x)$$

given that

$$y_h = c_1 e^{3x/2} + c_2 e^{-x/2}$$

Guess:

$$y_p = A\cos(2x) + B\sin(2x)$$

$$y_p' = -2A\sin(2x) + 2B\cos(2x)$$

$$y_p'' = -4A\cos(2x) - 4B\sin(2x)$$

Plug these into $4y'' - 4y' - 3y = \cos(2x)$.

We get:

$$4\overbrace{(-4A\cos(2x) - 4B\sin(2x))}^{y_p''}$$

$$-4\Big(\underbrace{-2A\sin(2x)+2B\cos(2x)}_{y_p'}\Big)$$

$$-3\big(A\cos(2x)+B\sin(2x)\big)\Big) = \cos(2x)$$

Reorganize:

$$\underbrace{(8A-19B)}_{0}\sin(2x)+\underbrace{(-19A-8B)}_{1}\cos(2x)=\cos(2x)$$

So, $\boxed{\begin{array}{l} 8A-19B=0 \\ -19A-8B=1 \end{array}} \leftarrow \boxed{\begin{array}{l} A=-19/425 \\ B=-8/425 \end{array}}$

So,

$$\boxed{y_p = -\frac{19}{425}\cos(2x)-\frac{8}{425}\sin(2x)}$$

General solution

$$y = y_h + y_p = c_1 e^{-3x/2} + c_2 e^{-x/2}$$

$$- \frac{19}{425} \cos(2x) - \frac{8}{425} \sin(2x)$$