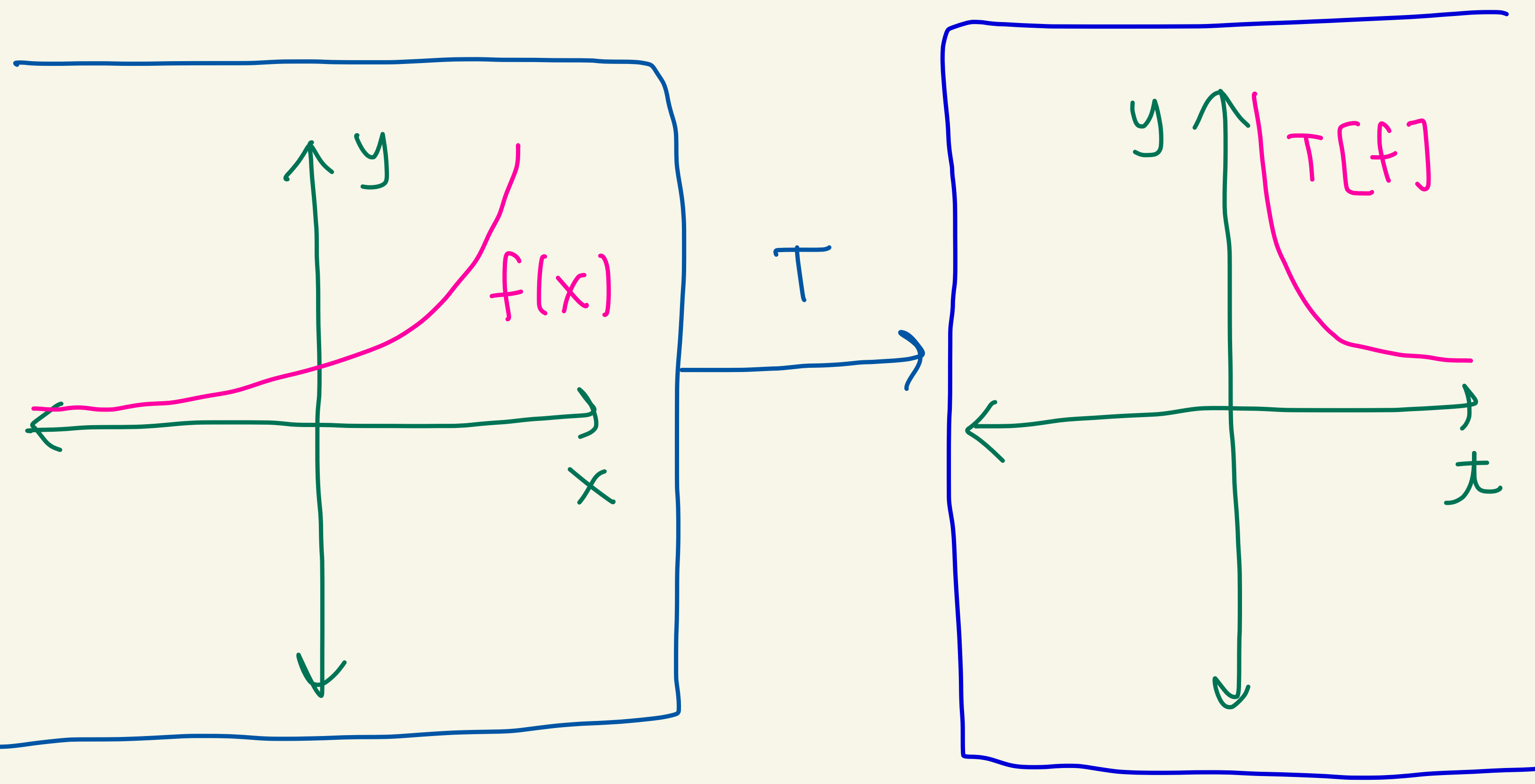
Math 2150-01
4/30/25

In mathematics an integral transform is a function T that takes another function f and transforms it into T[f] using the formula:

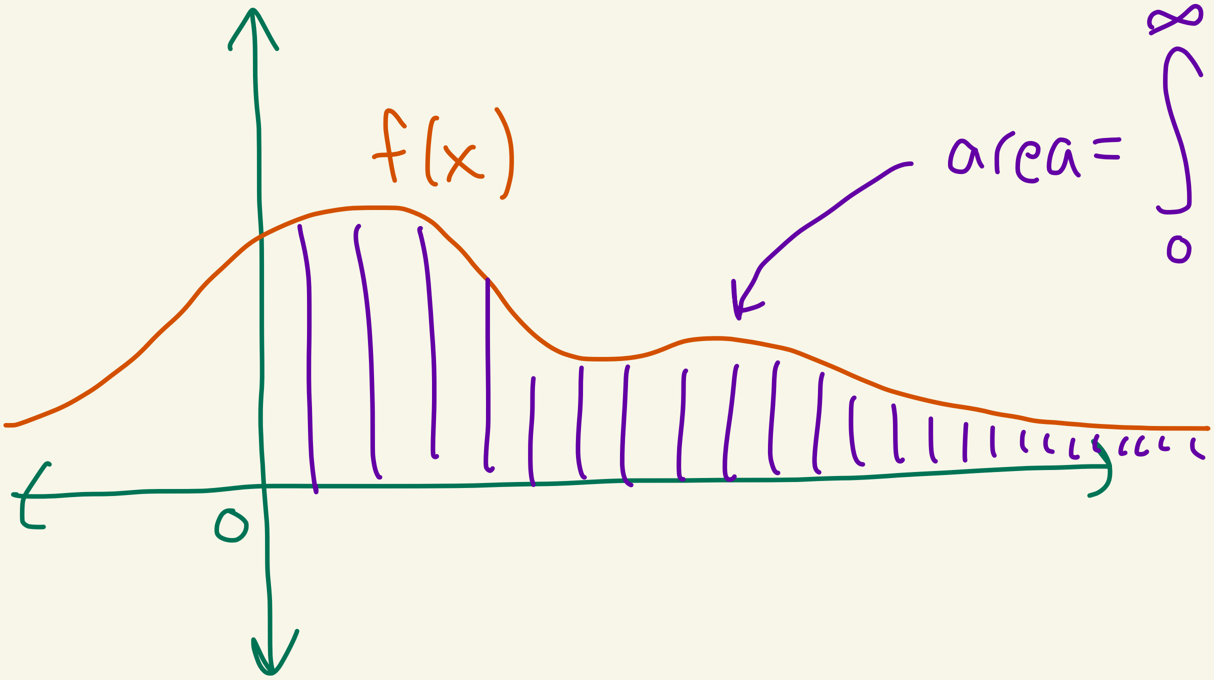
$$T[f](t) = \int_{x_1}^{x_2} \underbrace{K(x,t)}_{\text{the kernel of the transform}} f(x)\,dx$$



Examples

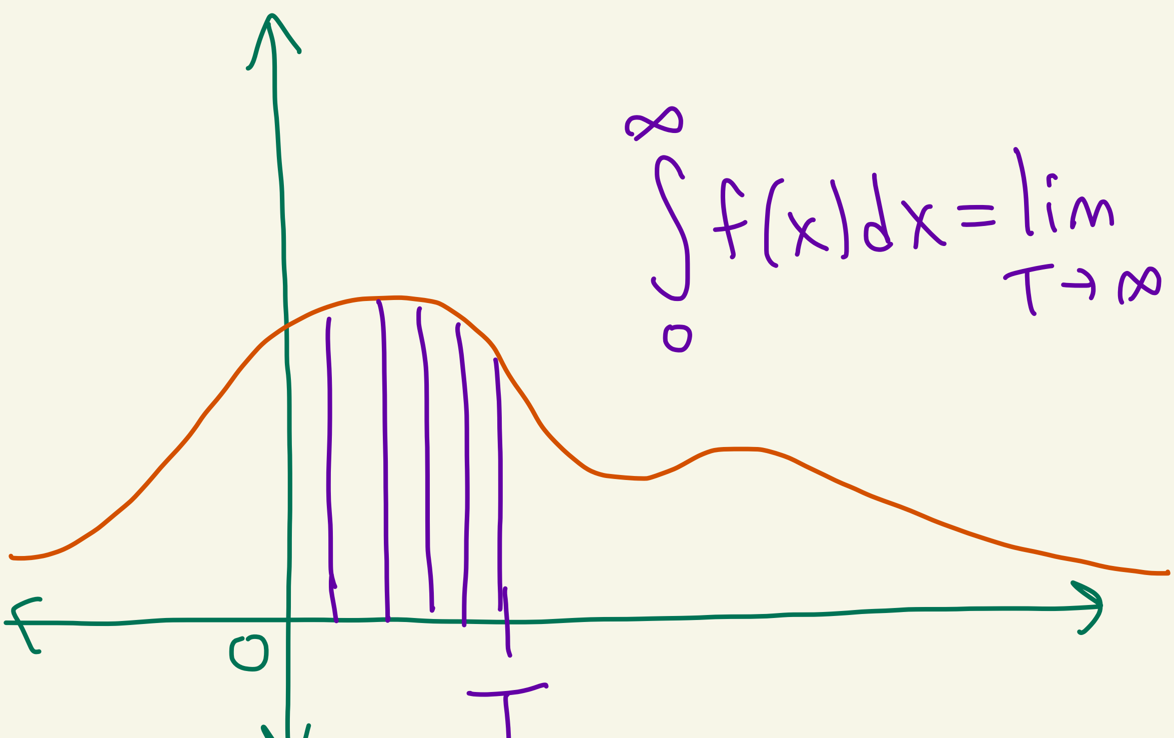
Name	Kernel
Fourier transform	$e^{-2\pi i t x}$
Laplace transform	e^{-tx}
Mellin transform	x^{t-1}
$\vdots$	$\vdots$

We need improper integrals for the Laplace transform.



$$\text{area} = \int_0^\infty f(x)\,dx$$

which is calculated



$$\int_0^\infty f(x)\,dx = \lim_{T\to\infty} \int_0^T f(x)\,dx$$

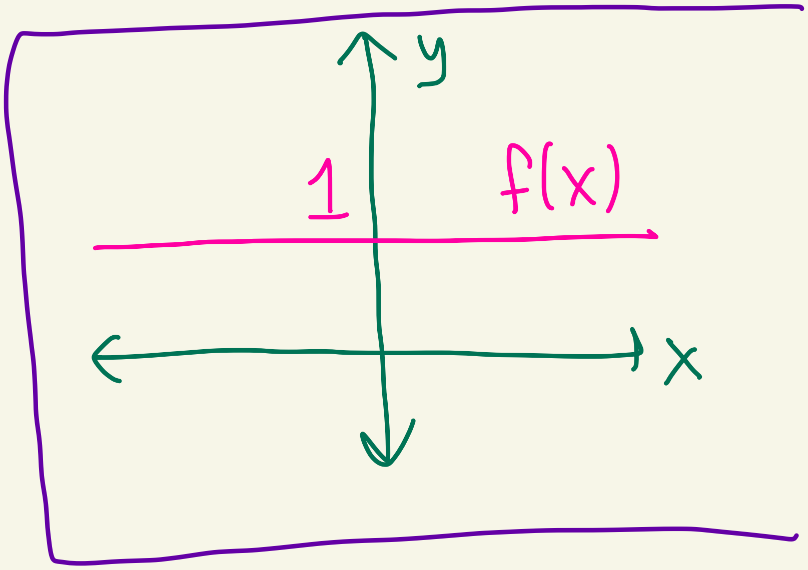
<u>Def:</u> Given a function $f(x)$ defined for $0 \le x < \infty$. Define the <u>Laplace transform</u> of f to be

$$\mathcal{L}[f] = \int_0^\infty \underbrace{e^{-tx}}_{\text{Kernel for Laplace transform}} f(x)\,dx$$

<u>Note:</u> $\mathcal{L}[f]$ is a function of t so we sometimes write

$$\underbrace{\mathcal{L}[f]}_{\text{name of function}}\underbrace{(t)}_{\text{plugging } t \text{ in}}$$

Ex: Let $f(x) = 1$ for all x.



Then,

$$\mathcal{L}[f] = \int_0^\infty e^{-tx} f(x)\,dx$$

$$\mathcal{L}[f] = \int_0^\infty e^{-tx}\,dx$$

$f(x)=1$

$\mathcal{L}[f]$ is a function of t.

For example,

we will see why below

$$\mathcal{L}[f](5) = \int_0^\infty e^{-5x}\,dx = \frac{1}{5}$$

$$\mathcal{L}[f](-7) = \int_0^\infty e^{-(-7)x}\,dx = \underbrace{\int_0^\infty e^{7x}\,dx}_{\text{this diverges}}$$

to ∞

It turns out $\mathcal{L}[f](t)$ is undefined / infinite when $t \leq 0$.

But when $t > 0$ we get

$$\mathcal{L}[f](t) = \int_0^\infty e^{-tx}\,dx$$

$$= \lim_{T\to\infty} \int_0^T e^{-tx}\,dx$$

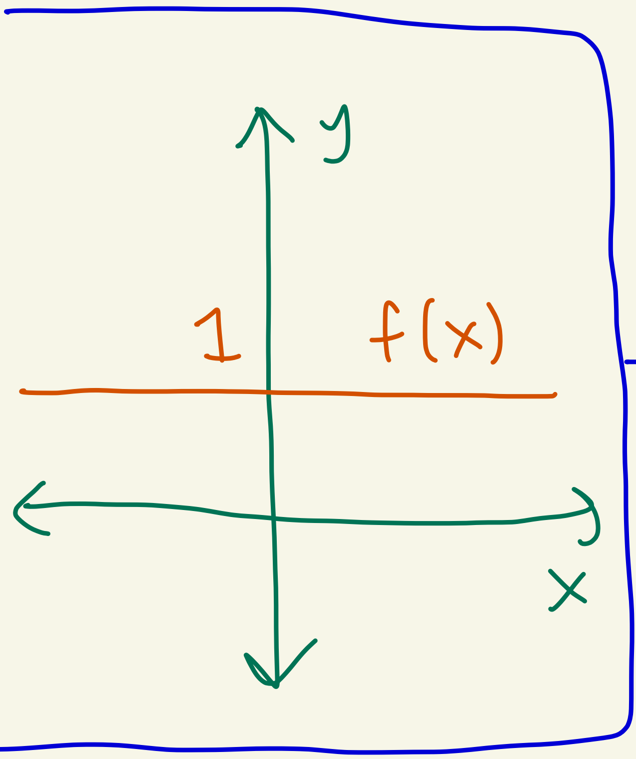
$$= \lim_{T\to\infty} \left[\frac{1}{-t} e^{-tx} \Big|_{x=0}^{T} \right]$$

$$= \lim_{T\to\infty} \left[\frac{1}{-t} e^{-tT} + \frac{1}{t} e^{0} \right]$$

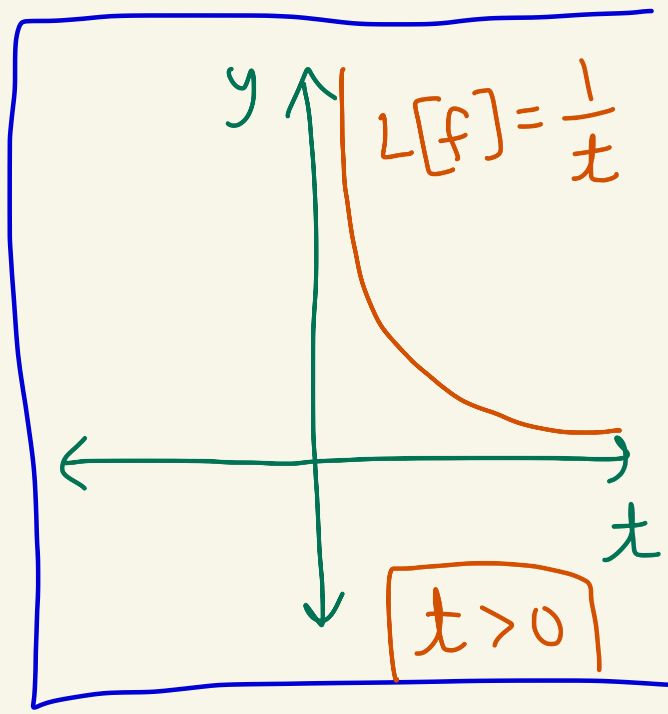
$$= \lim_{T\to\infty}\left[-\frac{1}{t}\cdot\frac{1}{e^{tT}}+\frac{1}{t}\right]$$

$\frac{1}{e^{tT}} \to 0$ as $T\to\infty$ because $t>0$

$$= \frac{1}{t}$$



$\xrightarrow{\mathcal{L}}$



Ex: Let $f(x) = e^{ax}$ where a is a constant.

If $t > a$, then

$$\mathcal{L}[f] = \int_0^\infty e^{-tx} f(x)\,dx$$

$$= \int_0^\infty e^{-tx} e^{ax}\,dx$$

$$= \int_0^\infty e^{(a-t)x}\,dx$$

$$= \lim_{T \to \infty} \int_0^T e^{(a-t)x}\,dx$$

$$= \lim_{T \to \infty} \left[\frac{1}{(a-t)} e^{(a-t)x} \Big|_{x=0}^{T} \right]$$

$$= \lim_{T\to\infty}\left[\frac{1}{(a-t)}\boxed{e^{(a-t)T}} - \frac{1}{(a-t)}e^{0}\right]$$

$t > a$

$0 > a-t$

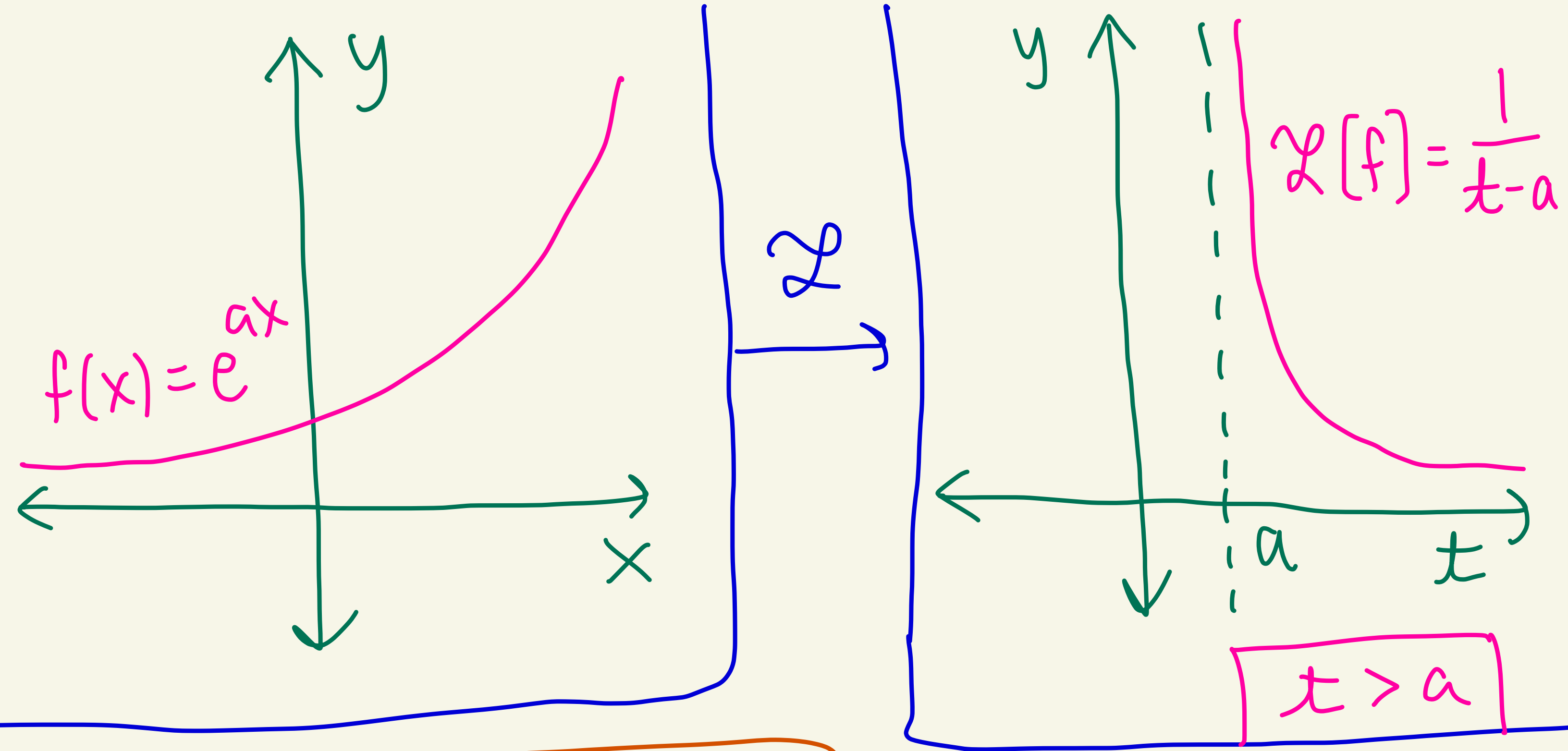
$a-t$ is negative

0

$$= 0 - \frac{1}{a-t} = \frac{1}{t-a}$$

If $t \leq a$, the Laplace transform would diverge to ∞.

Picture below



Some Laplace Transforms

① $\mathcal{L}[1] = \frac{1}{t}$ where $t > 0$

② $\mathcal{L}[x^n] = \frac{n!}{t^{n+1}}$ where $t > 0$, $n = 1, 2, 3, \ldots$

③ $\mathcal{L}[e^{ax}] = \frac{1}{t-a}$ where $t > a$

④ $\mathcal{L}[\sin(kx)] = \frac{k}{t^2+k^2}$ where $t > 0$

⑤ $\mathcal{L}[\cos(kx)] = \frac{t}{t^2+k^2}$ where $t > 0$

Theorem:

Suppose $\mathcal{L}[f]$ and $\mathcal{L}[g]$ both exist for $t > t_0$.

If c_1, c_2 are constants, then

$$\mathcal{L}[c_1 f + c_2 g] = c_1 \mathcal{L}[f] + c_2 \mathcal{L}[g]$$

for $t > t_0$.

Furthermore, if f, f', f'' are continuous for $0 \leq x < \infty$ and their Laplace transforms exist then

$$\mathcal{L}[f'] = t \cdot \mathcal{L}[f] - f(0)$$

$$\mathcal{L}[f''] = t^2 \cdot \mathcal{L}[f] - t f(0) - f'(0)$$