Math 2150-01

4/7/25

Topic 11 – Review of Power Series

<u>Def:</u> An <u>infinite sum</u> is a sum of the form

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + a_3 + \cdots$$

[This sum starts at $n=0$, but you can start n at any number]

The above sum <u>converges</u> to S if

$$\lim_{N\to\infty}\left[\sum_{n=0}^{N} a_n\right] = S$$

$$\lim_{N\to\infty}\left[a_0 + a_1 + \cdots + a_N\right] = S$$

If this is the case then we write $\sum_{n=0}^{\infty} a_n = S$.

If no such S exists, then the series <u>diverges</u>.

KACHOW!

95

Ex: Consider

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots$$

Let's calculate some partial sums.

N	$\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^N}$
1	$\frac{1}{2}=0{,}5$
2	$\frac{1}{2}+\frac{1}{2^2}=0{,}75$
3	$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}=0.875$
4	$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}=0.9375$
5	$0{,}96875$
$\vdots$	$\vdots$
50	$0{,}\underbrace{999\ldots99}_{15\ 9's}11182$
$\vdots$	$\vdots$
100	$0{,}\underbrace{999\ldots99}11139\ldots$

30 9's

In Calc II (Math 2120) you show that

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$$

$S = 1$ from def

Def: A power series is an infinite sum of the form

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \cdots$$

Where $x_0, a_0, a_1, a_2, \ldots$ are constants and x is a variable.

We say the power series is <u>centered</u> at x_0.

<u>Ex:</u> (Geometric sum)

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

$$= 1 + 1 \cdot (x-0) + 1 \cdot (x-0)^2 + \cdots$$

$a_0 = 1$ $\quad a_1 = 1$ $\quad x_0 = 0$ $\quad a_2 = 1$ $\quad x_0 = 0$

$x_0 = 0$ is center

Ex:

$$\sum_{n=0}^{\infty} \frac{1}{2^n}(x-3)^n$$

$$= 1 + \frac{1}{2}(x-3) + \frac{1}{2^2}(x-3)^2 + \dots$$

The center is $x_0 = 3$

Ex: (Back to geometric series...)

In Calculus you show that

$$\underbrace{\sum_{n=0}^{\infty} x^n}_{1+x+x^2+x^3+\dots} = \frac{1}{1-x} \quad \text{if } -1<x<1$$

The sum diverges otherwise.

For example,

$$\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n = 1+\frac{3}{4}+\left(\frac{3}{4}\right)^2+\left(\frac{3}{2}\right)^3+\cdots$$

$$= \frac{1}{1-3/4} = \frac{1}{1/4} = 4$$

$x = 3/4$

$-1 < x < 1$

And

$$\sum_{n=0}^{\infty}\pi^n = 1+\pi+\pi^2+\pi^3+\cdots$$

diverges to ∞ since $x=\pi$ does not satisfy $-1<x<1$.

You can think of the above as a function.

Let

$$f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

Then

$$f(0) = 1 + 0 + 0^2 + 0^3 + \cdots = 1$$

$$f\left(\tfrac{3}{4}\right) = 1 + \tfrac{3}{4} + \left(\tfrac{3}{4}\right)^2 + \cdots = 4$$

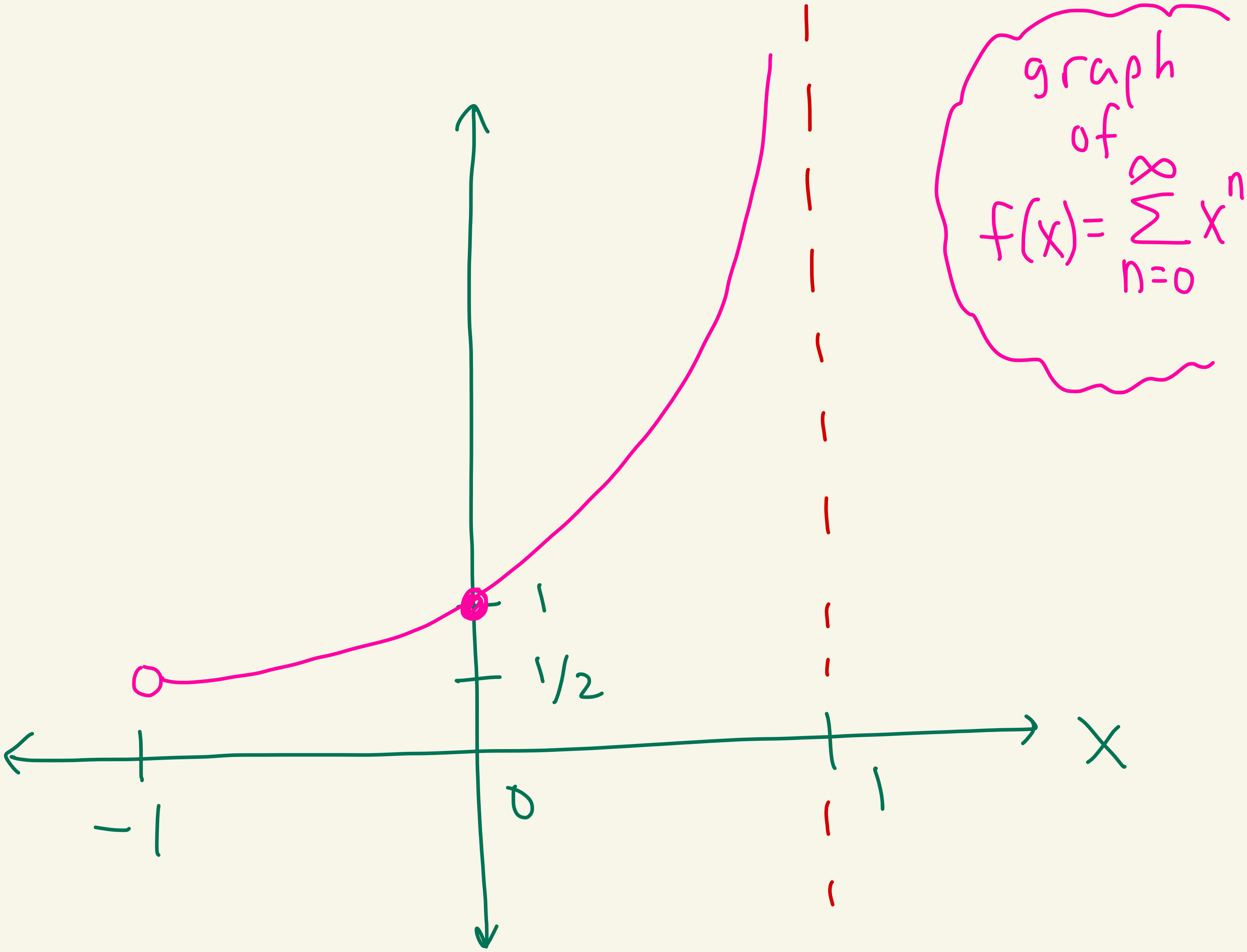
$$f\left(-\tfrac{1}{2}\right) = 1 + \left(-\tfrac{1}{2}\right) + \left(-\tfrac{1}{2}\right)^2 + \left(-\tfrac{1}{2}\right)^3 + \cdots$$

$$= \frac{1}{1-\left(-\frac{1}{2}\right)} = \frac{1}{3/2} = \frac{2}{3}$$

$$\boxed{\begin{gathered} x = -1/2 \\ -1 < x < 1 \\ 1 + x + x^2 + \cdots = \frac{1}{1-x} \end{gathered}}$$

$f(400) = 1 + 400 + 400^2 + 400^3 + \cdots$

diverges (x = 400 doesn't satisfy $-1 < x < 1$)

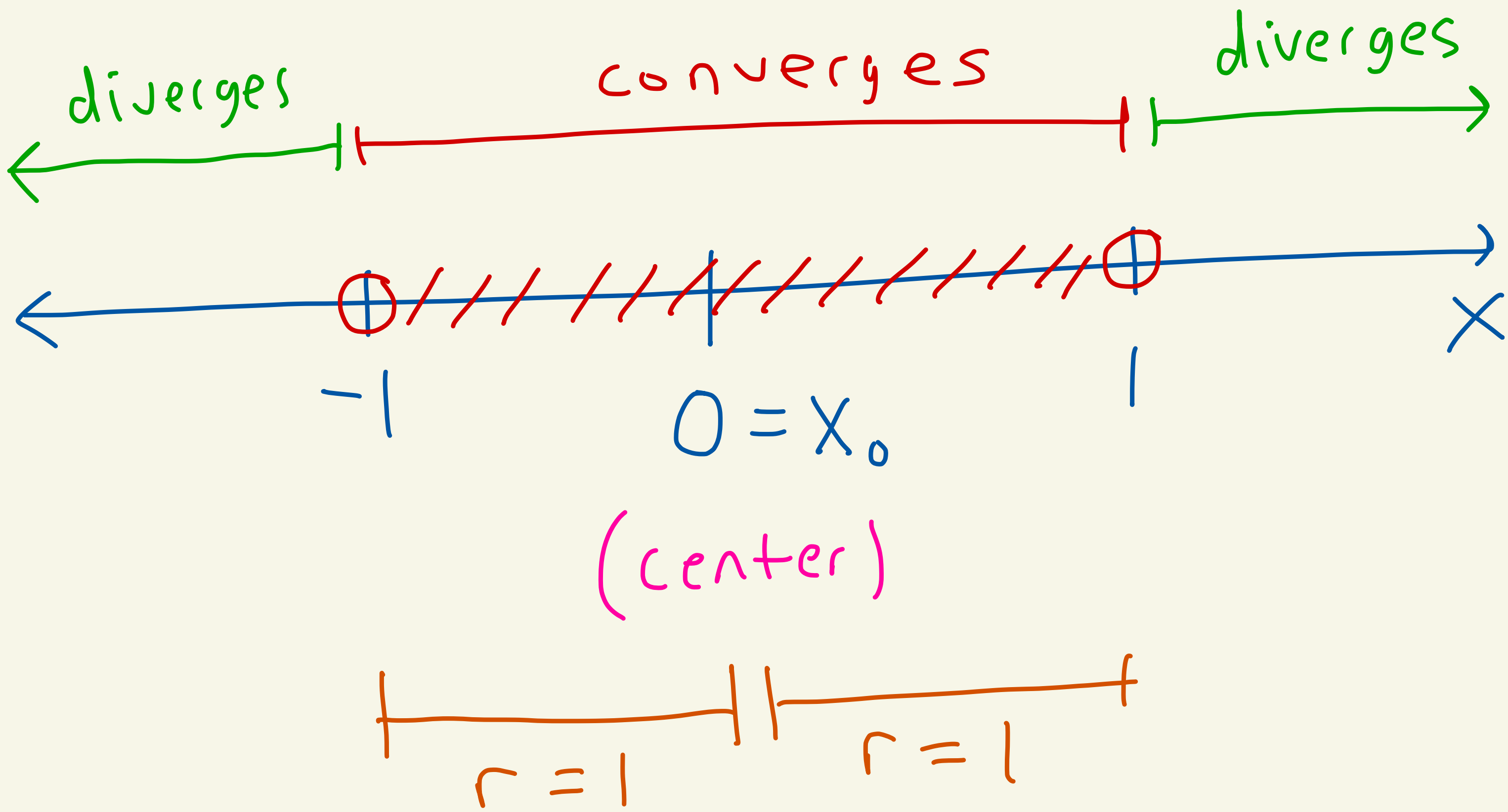


When $-1 < x < 1$, then $f(x) = \frac{1}{1-x}$

otherwise $f(x)$ diverges.

This is where

$$f(x) = \sum_{n=0}^{\infty} x^n \quad \text{converges:}$$



$r = 1$ is called the radius of convergence.

Ex: Recall from Calculus:

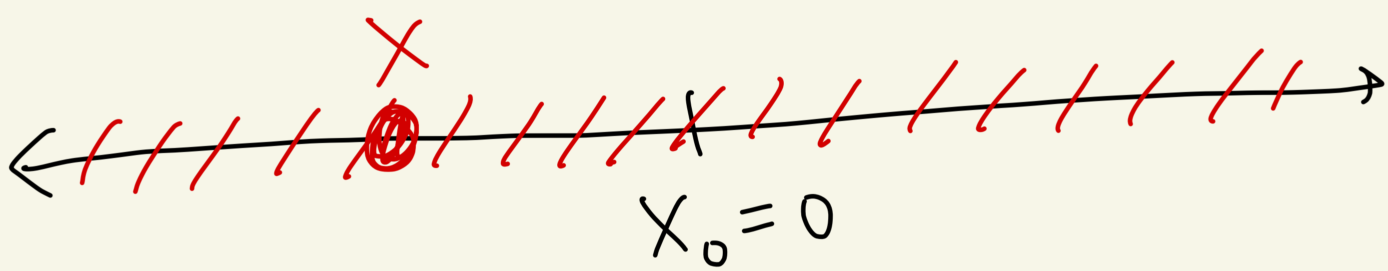
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$= \frac{1}{0!}x^0 + \frac{1}{1!}x^1 + \frac{1}{2!}x^2 + \frac{1}{3!}x + \dots$$

$$= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$$

$0! = 1$
$1! = 1$
$2! = 2 \cdot 1 = 2$
$3! = 3 \cdot 2 \cdot 1 = 6$
$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$
$\vdots$

This series converges for every x.
The center is $x_0 = 0$.



You can plug any x into sum.

The radius of convergence

is $r = \infty$.