Math 2150-01
4/9/25

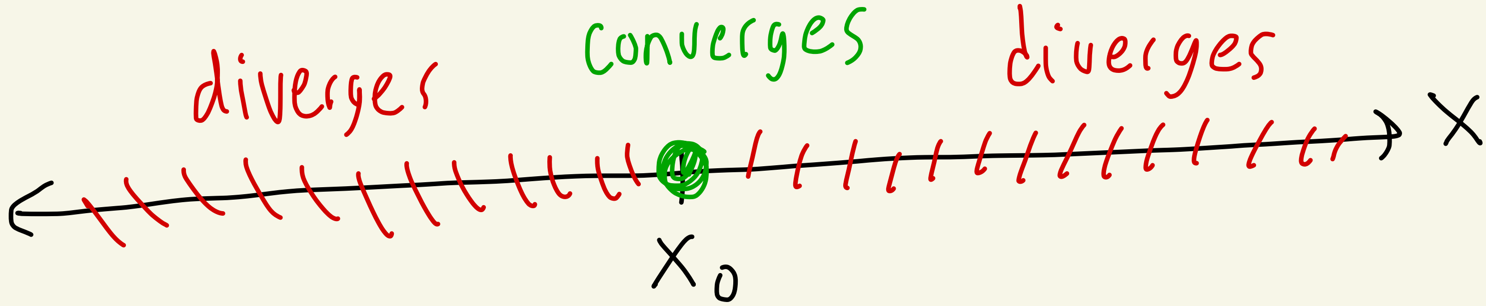
(Topic 11 continued...)

There are three possibilities for a power series

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \cdots$$

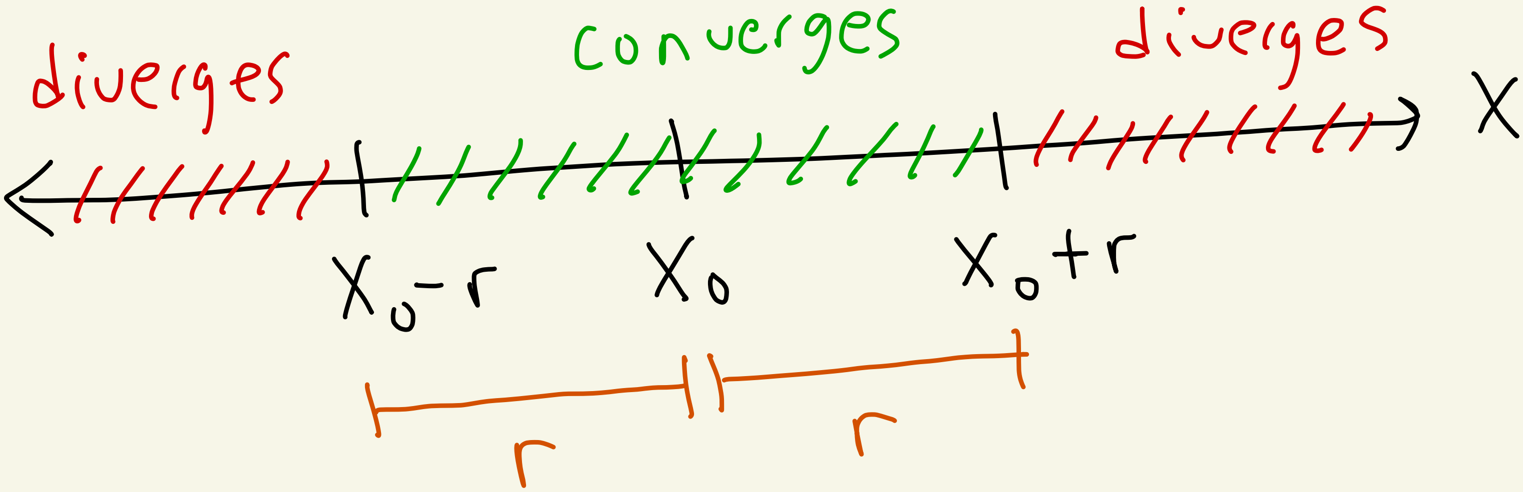
centered at x_0.

① The series only converges when $x = x_0$



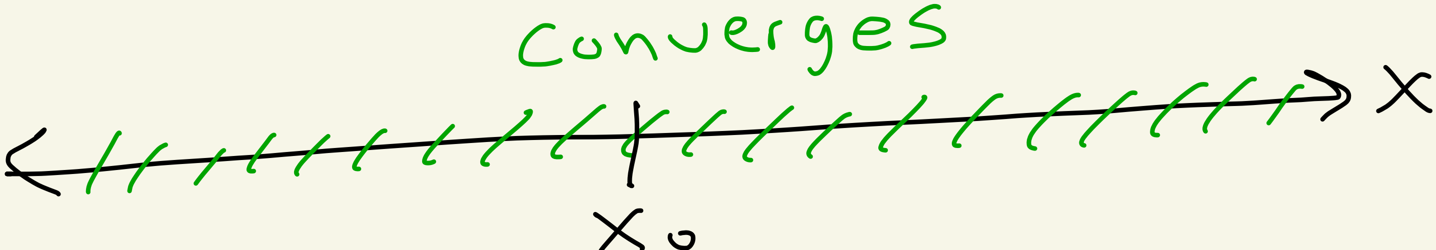
$r = 0$ is the radius of convergence

(2) there exists $r > 0$ where the series converges for all x with $x_0 - r < x < x_0 + r$



It can diverge or converge at the endpoints $x_0 - r$ and $x_0 + r$.
r is called the radius of convergence

(3) the series converges for all x.
$r = \infty$ is the radius of convergence.



Ex: Recall

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$

$$\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$$

These both converge for all x.
radius of convergence is $r = \infty$
Both are centered at $x_0 = 0$.

So, for example,

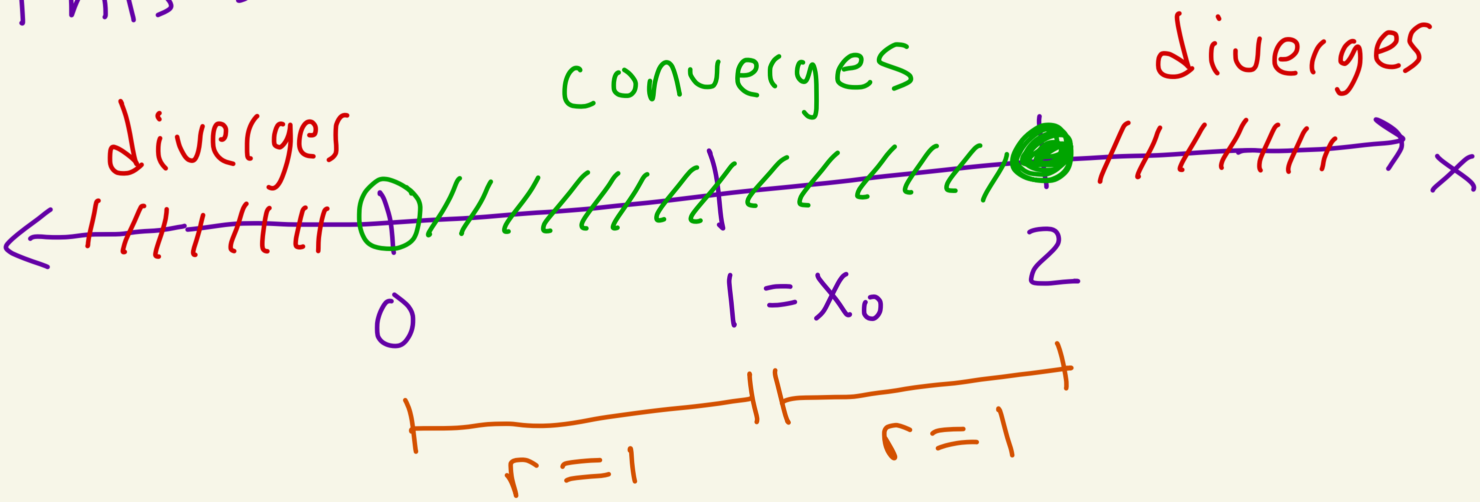
$$\sin(2) = 2 - \frac{1}{3!}(2)^3 + \frac{1}{5!}(2)^5 - \frac{1}{7!}(2)^7 + \cdots$$

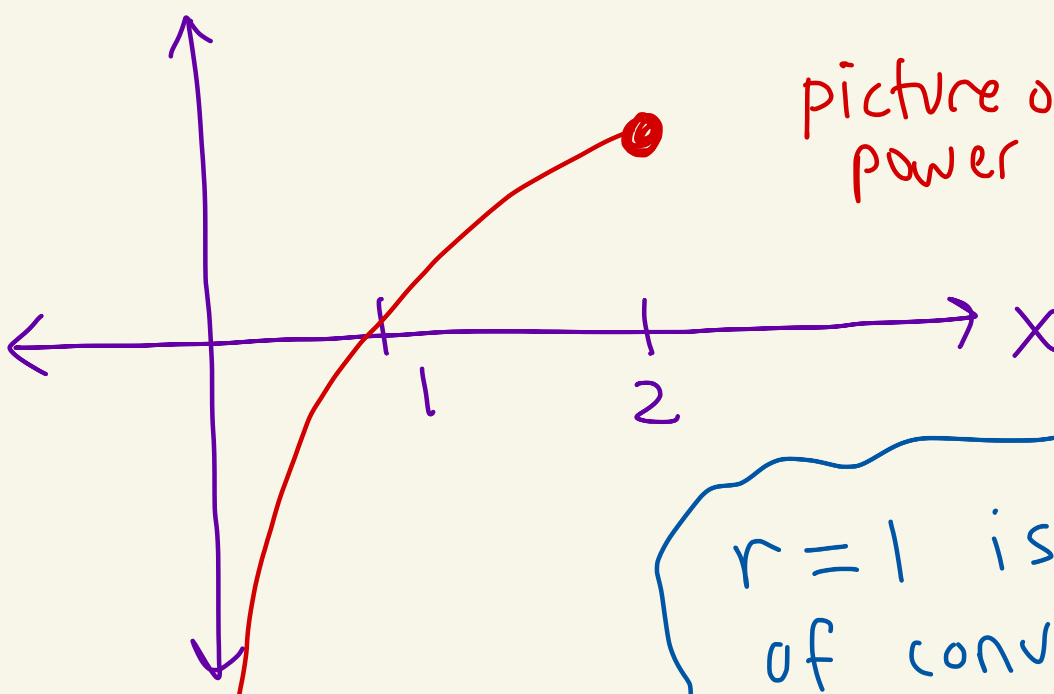
Ex: Recall

$$\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n \leftarrow \boxed{x_0 = 1}$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \cdots$$

This series converges for $0 < x \leq 2$.





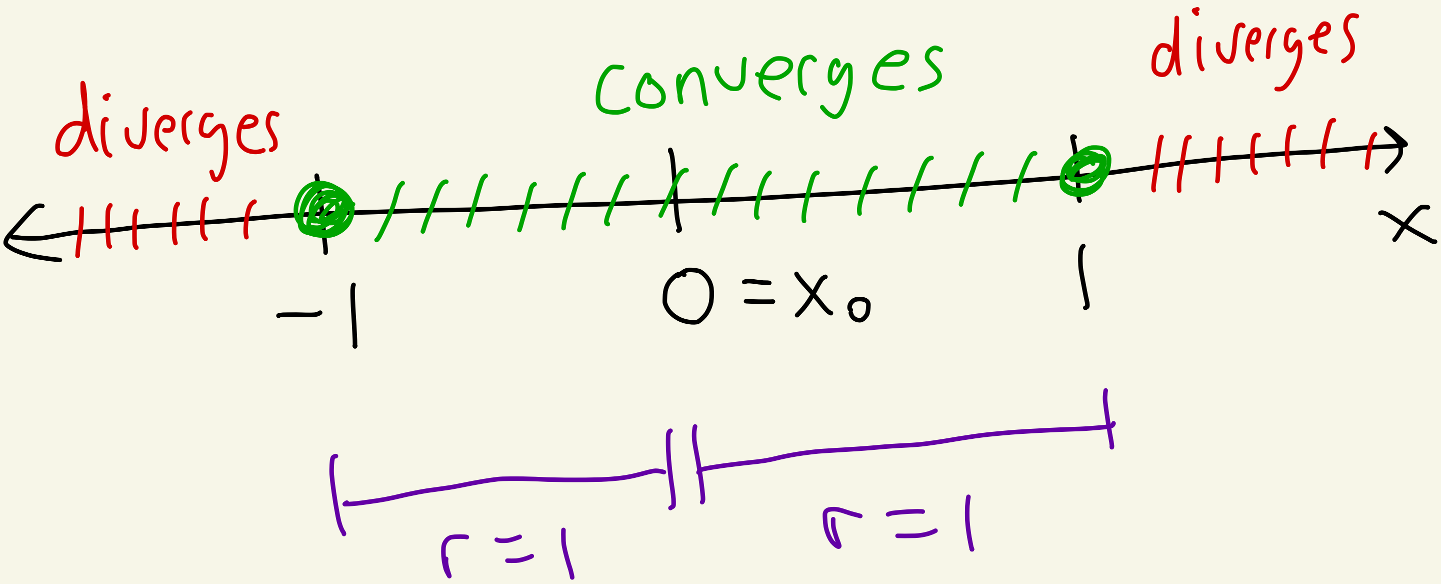
picture of power series

$r = 1$ is radius of convergence

Ex: If $-1 \leq x \leq 1$, then

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

$$= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$$



r = 1 is radius of convergence

Fun fact: Can use the above to approximate π.

$$\frac{\pi}{4} = \tan^{-1}(1) = \underbrace{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots}_{\text{plug } x=1 \text{ into power series above}}$$

Thus,

$$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \cdots$$

Theorem: (Taylor series)

If
$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

has radius of convergence $r > 0$.

Then,
$$a_n = \frac{f^{(n)}(x_0)}{n!}$$

So,

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0)$$
$$+ \frac{f''(x_0)}{2!}(x - x_0)^2$$
$$+ \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots$$

Ex: Find the power series for $f(x) = x^2$ centered at $x_0 = 2$.

$\boxed{\text{plug in center } x_0 = 2}$

$$f(x) = x^2 \longleftarrow f(2) = 4$$

$$f'(x) = 2x \longleftarrow f'(2) = 4$$

$$f''(x) = 2 \longleftarrow f''(2) = 2$$

$$f'''(x) = 0 \longleftarrow f'''(2) = 0$$

$$f''''(x) = 0 \longleftarrow f''''(2) = 0$$

$$\vdots \quad \vdots \qquad\qquad \vdots \quad \vdots$$

all 0

all 0

$$\underbrace{x^2}_{f(x)} = f(2) + f'(2)(x-2) + \frac{f''(2)}{2}(x-2)^2$$

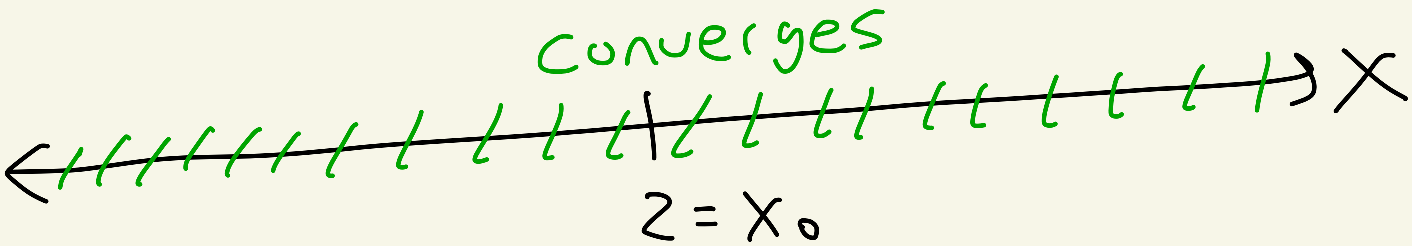
$$+ \underbrace{\boxed{\frac{f'''(2)}{6}(x-2)^3 + \cdots}}_{\text{zero}}$$

$$= 4 + 4(x-2) + \frac{2}{2}(x-2)^2$$

So,

$$\boxed{x^2 = 4 + 4(x-2) + (x-2)^2}$$

This will converge for every x since its just a finite sum.

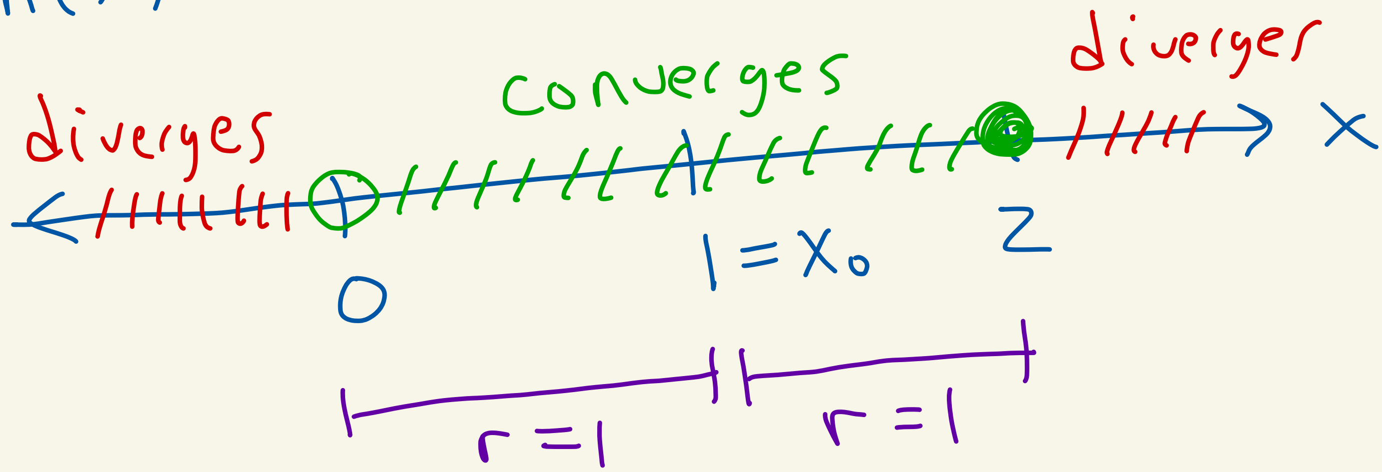


radius of convergence is $r = \infty$.

<u>Fact</u>: You can differentiate or integrate a function by differentiating or integrating it's power series term by term. This process doesn't change the radius of convergence.

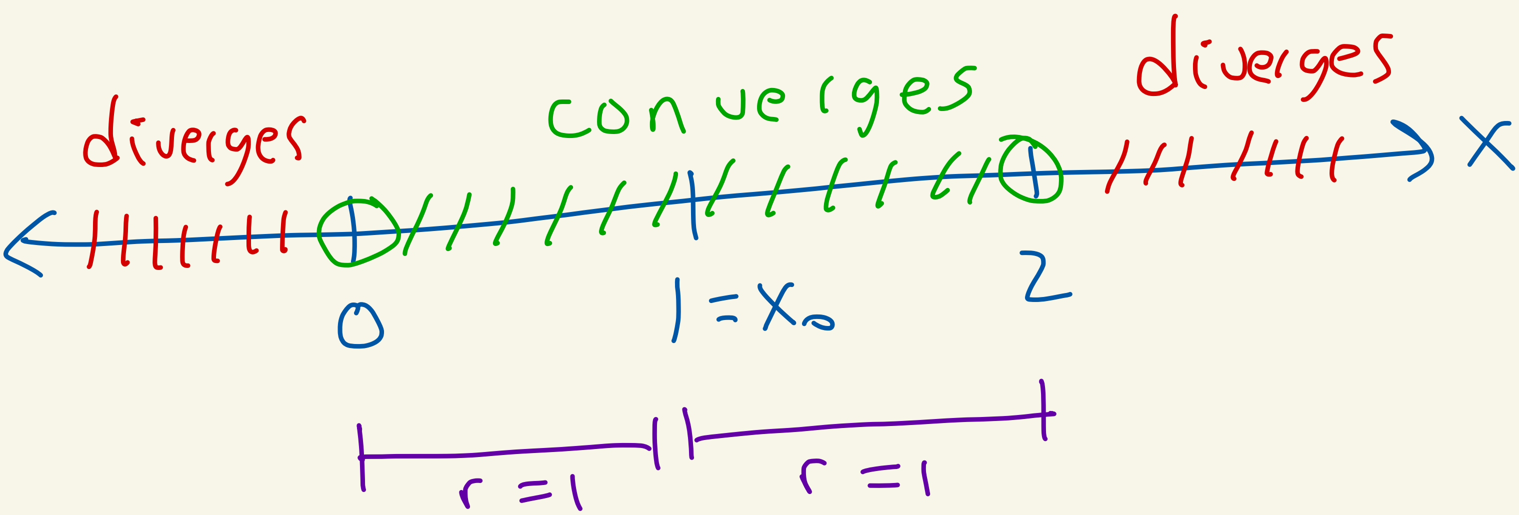
Ex:

$$\ln(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots$$



Let $0 < x < 2$,
Differentiate both sides of above

$$\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots$$



Same radius $r=1$, but endpoints not same convergence.