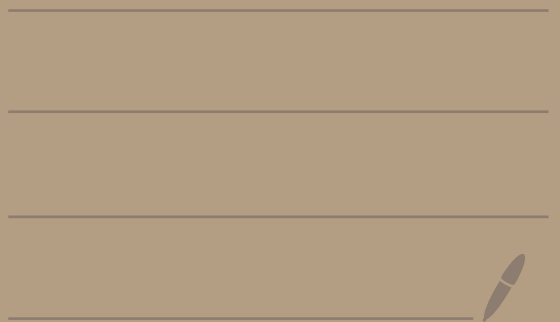


Math 2150-02

3/26/25



Topic 10 - Reduction of order

Suppose you have one solution y_1 to

$$y'' + a_1(x)y' + a_0(x)y = 0$$

(*)

on an interval I and

$$y_1(x) \neq 0 \text{ on } I.$$

Then one can find another solution using

$$y_2 = y_1 \int \frac{e^{-\int a_1(x) dx}}{y_1^2} dx$$

and you will get that y_1, y_2 are linearly independent giving

that

$$y_h = c_1 y_1 + c_2 y_2$$

is the general solution to $(*)$.

[Derivation in notes online]

Ex: Consider

$$(x^2 + 1)y'' - 2xy' + 2y = 0$$

on $I = (0, \infty)$

One solution to the above
is $y_1 = x$.

Let's find the general
solution y_h to our ODE

Divide by $x^2 + 1$ to put the

ODE into the form we want:

$$y'' - \frac{2x}{x^2+1} y' + \frac{2}{x^2+1} y = 0$$

Use the formula:

$$a_1(x)$$

$$y_2 = y_1 \int \frac{e^{-\int a_1(x) dx}}{y_1^2} dx$$

$$= x \int \frac{e^{-\int \frac{-2x}{x^2+1} dx}}{x^2} dx$$

$$= x \int \frac{e^{\int \frac{2x}{x^2+1} dx}}{x^2} dx$$

$$= x \int \frac{e^{\ln(x^2+1)}}{x^2} dx$$



$$\int \frac{2x}{x^2+1} dx = \int \frac{1}{u} du = \ln|u|$$

$$= \ln|x^2+1|$$

$$= \ln(x^2+1)$$

$$u = x^2 + 1$$

$$du = 2x dx$$

$$= x \int \frac{x^2+1}{x^2} dx$$



$$e^{\ln(A)} = A$$

$$= x \int \left(\frac{x^2}{x^2} + \frac{1}{x^2} \right) dx$$

$$= x \int (1 + x^{-2}) dx$$

$$= x \left(x + \frac{x^{-1}}{-1} \right)$$

$$= x \left(x - \frac{1}{x} \right)$$

$$= x^2 - 1$$

Thus, $y_1 = x$ and $y_2 = x^2 - 1$.

The general solution is

$$\begin{aligned} y_h &= c_1 y_1 + c_2 y_2 \\ &= c_1 x + c_2 (x^2 - 1) \end{aligned}$$

HW 10 - #1(a)

Given that $y_1 = x^4$ is a solution to

$$x^2 y'' - 7x y' + 16y = 0$$

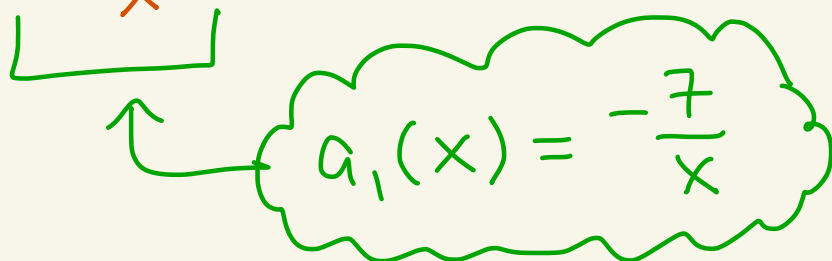
on $I = (0, \infty)$.

Find another solution y_2
and the general solution y_h

Need to divide by x^2 to
get a 1 in front of y'' .

We get:

$$y'' - \frac{7}{x} y' + \frac{16}{x^2} y = 0$$


$$a_1(x) = -\frac{7}{x}$$

We have

$$y_2 = y_1 \int \frac{e^{-\int a_1(x) dx}}{y_1^2} dx$$

$$= x^4 \int \frac{e^{-\int -\frac{7}{x} dx}}{(x^4)^2} dx$$

$$= x^4 \int \frac{e^{\int \frac{7}{x} dx}}{x^8} dx$$

$$= x^4 \int \frac{e^{7 \ln|x|}}{x^8} dx$$

$$= x^4 \int \frac{e^{7 \ln(x)}}{x^8} dx$$



$$I = (0, \infty)$$

$$x > 0$$

$$|x| = x$$

$$= x^4 \int \frac{e^{\ln(x^7)}}{x^8} dx$$

$$A \ln(B) = \ln(B^A)$$

$$= x^4 \int \frac{x^7}{x^8} dx$$

$$e^{\ln(A)} = A$$

$$= x^4 \int \frac{1}{x} dx$$

$$= x^4 \ln|x|$$

$$= x^4 \ln(x)$$

$$I = (0, \infty)$$

$$x > 0$$

Thus, the general solution to the ODE is

$$\begin{aligned} y_h &= c_1 y_1 + c_2 y_2 \\ &= c_1 x^4 + c_2 x^4 \ln(x) \end{aligned}$$