Math 2150-02
3/26/25

Topic 10 - Reduction of order

Suppose you have one solution y_1 to

$$\boxed{y'' + a_1(x)y' + a_0(x)y = 0} \quad (*)$$

on an interval I and $y_1(x) \neq 0$ on I.

Then one can find another solution using

$$\boxed{y_2 = y_1 \int \frac{e^{-\int a_1(x)dx}}{y_1^2}\,dx}$$

and you will get that y_1, y_2 are linearly independent giving that

$$y_h = c_1 y_1 + c_2 y_2$$

is the general solution to $(*)$.

[Derivation in notes online]

Ex: Consider

$$(x^2+1)y'' - 2xy' + 2y = 0$$

on $I = (0, \infty)$

One solution to the above is $y_1 = x$.

Let's find the general solution y_h to our ODE

Divide by x^2+1 to put the

ODE into the form we want:

$$y'' \boxed{- \frac{2x}{x^2+1}} y' + \frac{2}{x^2+1} y = 0$$

$a_1(x)$

Use the formula:

$$y_2 = y_1 \int \frac{e^{-\int a_1(x)\,dx}}{y_1^2}\,dx$$

$$= x \int \frac{e^{-\int \frac{-2x}{x^2+1}dx}}{x^2}\,dx$$

$$= x \int \frac{e^{\int \frac{2x}{x^2+1}dx}}{x^2}\,dx$$

$$= x \int \frac{e^{\ln(x^2+1)}}{x^2}\,dx$$

$$\int \frac{2x}{x^2+1}\,dx = \int \frac{1}{u}\,du = \ln|u|$$

$$= \ln|x^2+1|$$

$$= \ln(x^2+1)$$

$$u = x^2+1$$

$$du = 2x\,dx$$

$$= x \int \frac{x^2+1}{x^2}\,dx$$

$$e^{\ln(A)} = A$$

$$= x \int \left(\frac{x^2}{x^2} + \frac{1}{x^2}\right) dx$$

$$= x \int (1 + x^{-2})\,dx$$

$$= x\left(x + \frac{x^{-1}}{-1}\right)$$

$$= x\left(x - \frac{1}{x}\right)$$

$$= x^2 - 1$$

Thus, $y_1 = x$ and $y_2 = x^2 - 1$.

The general solution is

$$\boxed{\begin{aligned} y_h &= c_1 y_1 + c_2 y_2 \\ &= c_1 x + c_2 (x^2 - 1) \end{aligned}}$$

HW 10 – #1(a)

Given that $y_1 = x^4$ is a solution to

$$x^2 y'' - 7xy' + 16y = 0$$

on $I = (0, \infty)$.

Find another solution y_2 and the general solution y_h

Need to divide by x^2 to get a 1 in front of y''.

We get:

$$y'' \underbrace{- \frac{7}{x}}_{a_1(x) = -\frac{7}{x}} y' + \frac{16}{x^2} y = 0$$

We have

$$y_2 = y_1 \int \frac{e^{-\int a_1(x)\,dx}}{y_1^2}\,dx$$

$$= x^4 \int \frac{e^{-\int -\frac{7}{x}\,dx}}{(x^4)^2}\,dx$$

$$= x^4 \int \frac{e^{\int \frac{7}{x}\,dx}}{x^8}\,dx$$

$$= x^4 \int \frac{e^{7\ln|x|}}{x^8}\,dx$$

$$= x^4 \int \frac{e^{7\ln(x)}}{x^8}\,dx$$

$I=(0,\infty)$
$x>0$
$|x|=x$

$$= x^4 \int \frac{e^{\ln(x^7)}}{x^8}\,dx$$

$A\ln(B)=\ln(B^A)$

$$= x^4 \int \frac{x^7}{x^8}\,dx$$

$e^{\ln(A)}=A$

$$= x^4 \int \frac{1}{x}\,dx$$

$$= x^4 \ln|x|$$

$I=(0,\infty)$
$x>0$

$$= x^4 \ln(x)$$

Thus, the general solution to the ODE is

$$\boxed{\begin{aligned} y_h &= c_1 y_1 + c_2 y_2 \\ &= c_1 x^4 + c_2 x^4 \ln(x) \end{aligned}}$$