Math 2150-02
4/16/25

Ex: Consider

$$\boxed{\begin{aligned} &y' - 2xy = 0 \\ &y(0) = 1 \end{aligned}}$$

$x_0 = 0$ for power series

$$y' \boxed{-2x}\, y = \boxed{0}$$

power series centered at $x_0 = 0$

$$a_1(x) = -2x = 0 - 2x + 0x^2 + 0x^3 + \dots$$

$$a_0(x) = 0 = 0 + 0x + 0x^2 + 0x^3 + \dots$$

The coefficients $a_1(x) = -2x$ and $a_0(x) = 0$ are analytic at $x_0 = 0$ because they have a power series at $x_0 = 0$ and they both have radius of convergence $r = \infty$.

Thus, by the theorem there is a power series solution

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \cdots$$

with radius of convergence $r = \infty$

To fill in this power series we need $y^{(n)}(0)$.

We use

$$\boxed{\begin{aligned} y' - 2xy &= 0 \\ y(0) &= 1 \end{aligned}} \leftarrow y' = 2xy$$

We have $\boxed{y(0) = 1}$

Also, $y'(0) = 2(0)[y(0)]$

$= 2(0)(1) = 0$

So, $\boxed{y'(0) = 0}$

To find $y''(0)$ differentiate $y' = 2xy$

to get $y'' = 2y + 2xy'$

So, $y''(0) = 2[y(0)] + 2(0)[y'(0)]$

$= 2(1) + 2(0)(0)$

$= 2$

Thus, $\boxed{y''(0) = 2}$

Now differentiate $y'' = 2y + 2xy'$

to get $y''' = 2y' + 2y' + 2xy''$

Thus, $y'''(0) = 2[y'(0)] + 2[y'(0)]$

$$+ 2(0)\left[y''(0)\right]$$

$$= 2[0] + 2[0] + 2(0)(2)$$

$$= 0$$

So, $\boxed{y'''(0) = 0}$

Differentiate $y''' = 4y' + 2xy''$

to get $y'''' = 4y'' + 2y'' + 2xy'''$

We get

$$y''''(0) = 4\left[y''(0)\right] + 2\left[y''(0)\right]$$

$$+ 2(0)\left[y'''(0)\right]$$

$$= 4[2] + 2[2] + 0$$

$$= 12$$

Thus, $\boxed{y''''(0) = 12}$

So,

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2$$
$$+ \frac{y'''(0)}{3!}x^3 + \frac{y'''(0)}{4!}x^4 + \cdots$$

$$y(x) = 1 + 0x + \frac{2}{2!}x^2 + \frac{0}{3!}x^3$$
$$+ \frac{12}{4!}x^4 + \cdots$$

$$\boxed{y(x) = 1 + x^2 + \frac{1}{2}x^4 + \cdots}$$

radius of convergence $r = \infty$

Side note

If you use previous methods you'd get

$$y(x) = e^{x^2} = 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \cdots$$

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$$

Ex: Consider

$$\boxed{\begin{array}{l} y'' + x^2 y' - (x-1)y = \ln(x) \\ y'(1) = 0, \quad y(1) = 0 \end{array}}$$

$\{x_0 = 1\}$

Coefficients:

$$\left.\begin{array}{l} x^2 = 1 + 2(x-1) + (x-1)^2 + 0(x-1)^3 + \dots \\ -(x-1) = 0 - 1\cdot(x-1) + 0\cdot(x-1)^2 + \dots \end{array}\right] r = \infty$$

$$\ln(x) = -(x-1) + \tfrac{1}{2}(x-1)^2 + \dots \Big] \; r = 1$$

Theorem says there will be a power series solution

$$y(x) = y(1) + y'(1)(x-1) + \frac{y''(1)}{2!}(x-1)^2 + \frac{y'''(1)}{3!}(x-1)^3 + \dots$$

and it will have radius of convergence at least $r=1$.

Now let's find the first few terms of the power series solution.

$$\boxed{\begin{gathered} y'' + x^2 y' - (x-1)y = \ln(x) \\ y'(1)=0,\ y(1)=0 \end{gathered}}$$

$$\downarrow$$

$$\boxed{\begin{gathered} y(1)=0 \\ y'(1)=0 \end{gathered}}$$

$$\downarrow$$

$$y'' = -x^2 y' + (x-1)y + \ln(x)$$

$$y''(1) = -(1)^2\left[y'(1)\right] + (1-1)\left[y(1)\right] + \ln(1)$$

$$= -1\cdot\left[0\right] + (0)(0) + 0 = 0$$

$y''(1) = 0$

$$y''' = -2xy' - x^2y'' + 1 \cdot y + (x-1)y' + \frac{1}{x}$$

$$y'''(1) = -2(1)[y'(1)] - (1)^2[y''(1)] + y(1) + (1-1)[y'(1)] + \frac{1}{1}$$

$$= -2[0] - (1)[0] + [0] + 0 + \frac{1}{1}$$

$$= 1$$

$y'''(1) = 1$

If you did this again you'd get $y''''(1) = -3$

Thus,

$$y(x) = 0 + 0(x-1) + \frac{0}{2!}(x-1)^2 + \frac{1}{3!}(x-1)^3 + \frac{-3}{4!}(x-1)^4 + \cdots$$

$$\boxed{y(x) = \frac{1}{6}(x-1)^3 - \frac{1}{8}(x-1)^4 + \cdots}$$

will radius of convergence at least $r = 1$