Math 2150-02
4/21/25

Practice Test 1

<u>HW 8 - 1(c)</u> Solve

$$\frac{1}{4}y'' + y' + y = x^2 - 2x$$

Step 1: Find y_h

$$\frac{1}{4}y'' + y' + y = 0$$

$$\frac{1}{4}r^2 + r + 1 = 0$$

$\times 4$

$$r^2 + 4r + 4 = 0$$

$$r = \frac{-4 \pm \sqrt{4^2 - 4(1)(4)}}{2(1)} = \frac{-4 \pm \sqrt{0}}{2} = -2$$

$$y_h = c_1 e^{-2x} + c_2 x e^{-2x}$$

Step 2: Find y_p for

$$\frac{1}{4}y'' + y' + y = \boxed{x^2 - 2x}$$

Use table

Guess $y_p = Ax^2 + Bx + C$

$$y_p' = 2Ax + B$$

$$y_p'' = 2A$$

Plug these into $\frac{1}{4}y'' + y' + y = x^2 - 2x$.

We get

$$\frac{1}{4}\underbrace{(2A)}_{y_p''} + \underbrace{(2Ax + B)}_{y_p'} + \underbrace{(Ax^2 + Bx + C)}_{y_p} = x^2 - 2x$$

We get

$$\underbrace{A}_{1}x^2 + \underbrace{(2A + B)}_{-2}x + \underbrace{\left(\tfrac{1}{2}A + B + C\right)}_{0} = x^2 - 2x$$

So,

$$\boxed{\begin{aligned} A &= 1 \\ 2A+B &= -2 \\ \tfrac{1}{2}A+B+C &= 0 \end{aligned}}$$

$\leftarrow \boxed{A=1}$

$\leftarrow \boxed{\begin{aligned} 2(1)+B &= -2 \\ B &= -4 \end{aligned}}$

$\leftarrow \boxed{\begin{aligned} \tfrac{1}{2}(1)+(-4)+C &= 0 \\ C &= 7/2 \end{aligned}}$

So, $\boxed{y_p = Ax^2+Bx+C = x^2-4x+7/2}$

Thus, the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= c_1 e^{-2x} + c_2 x e^{-2x} + x^2 - 4x + 7/2 \end{aligned}$$

HW 9 - 1(b)

Suppose you are given that the solution to $y'' + y = 0$ is $y_h = c_1 \cos(x) + c_2 \sin(x)$.

Use variation of parameters to find y_p for

$$y'' + y = \sin(x).$$

Then state the general solution.

$y_1 = \cos(x)$

$y_2 = \sin(x)$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

$$= \cos(x)\cos(x) - (-\sin(x))\sin(x)$$

$$= \cos^2(x) + \sin^2(x) = 1$$

Then,

$$v_1 = \int \frac{-y_2 b(x)}{W(y_1, y_2)}\, dx$$

$$= \int \frac{-\sin(x)\sin(x)}{1}\, dx$$

$$= -\int \sin^2(x)\, dx$$

$$= -\int \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right) dx$$

$$= -\left(\frac{1}{2}x - \frac{1}{2}\left(\frac{1}{2}\sin(2x)\right)\right)$$

$$= -\frac{1}{2}x + \frac{1}{4}\sin(2x)$$

$$v_2 = \int \frac{y_1 b(x)}{W(y_1, y_2)}\,dx = \int \frac{\cos(x)\sin(x)}{1}\,dx$$

$$= \int \underbrace{\underbrace{\cos(x)}\,\underbrace{\sin(x)}_{u}\,\underbrace{dx}}_{du}$$

$$u = \sin(x)$$
$$du = \cos(x)\,dx$$

$$= \int u\,du = \frac{1}{2}u^2 = \frac{1}{2}\sin^2(x)$$

Thus,

$$y_p = v_1 y_1 + v_2 y_2$$
$$= \left(-\frac{1}{2}x + \frac{1}{4}\sin(2x)\right)\cos(x)$$
$$+ \left(\frac{1}{2}\sin^2(x)\right)\sin(x)$$

The general solution is

$$y = y_h + y_p$$

$$= c_1\cos(x) + c_2\sin(x) + \left(-\tfrac{1}{2}x + \tfrac{1}{4}\sin(2x)\right)\cos(x) + \left(\tfrac{1}{2}\sin^2(x)\right)\sin(x)$$

HW 10
1(a)

Given $y_1 = x^4$ solves

$$x^2y'' - 7xy' + 16y = 0$$

on $I = (0, \infty)$.

(a) Find another linearly independent solution y_2

(b) State the general solution

Divide by x^2 to get

$$y'' - \frac{7}{x} y' + \frac{16}{x^2} y = 0$$

$$a_1(x) = -\frac{7}{x}$$

Then

$$y_2 = y_1 \int \frac{e^{-\int a_1(x)\,dx}}{y_1^2}\,dx$$

$$= x^4 \int \frac{e^{-\int -\frac{7}{x}\,dx}}{(x^4)^2}\,dx$$

$$= x^4 \int \frac{e^{7\int \frac{1}{x}\,dx}}{x^8}\,dx$$

$$= x^4 \int \frac{e^{7\ln|x|}}{x^8}\,dx$$

$\boxed{\begin{array}{l} I=(0,\infty) \\ x>0 \\ |x|=x \end{array}}$

$$= x^4 \int \frac{e^{7\ln(x)}}{x^8}\,dx$$

$$= x^4 \int \frac{e^{\ln(x^7)}}{x^8}\,dx$$

$\boxed{\begin{array}{l} A\ln(B) \\ =\ln(B^A) \end{array}}$

$$= x^4 \int \frac{x^7}{x^8}\,dx$$

$\boxed{e^{\ln(A)} = A}$

$$= x^4 \int \frac{1}{x}\,dx$$

$$= x^4 \ln(x)$$

$\boxed{x>0}$

So, $y_1 = x^4$, $y_2 = x^4\ln(x)$

general solution

$$y_h = c_1 y_1 + c_2 y_2$$

$$= c_1 x^4 + c_2 x^4 \ln(x)$$

$\boxed{\text{HW 11 } 1(a)}$ Find the power series for $f(x) = x^3 + x$ centered at $x_0 = 1$.

$f(x) = x^3 + x \longleftarrow f(1) = 1^3 + 1 = 2$

$f'(x) = 3x^2 + 1 \longleftarrow f'(1) = 3(1)^2 + 1 = 4$

$f''(x) = 6x \longleftarrow f''(1) = 6(1) = 6$

$f'''(x) = 6 \longleftarrow f'''(1) = 6$

$f''''(x) = 0$ 0 after this point $\longleftarrow$ all zero at $x_0 = 1$

$\vdots \quad \vdots$

$$f(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \frac{f''''(1)}{4!}(x-1)^4 + \dots$$

$$\underbrace{x^3 + x}_{f(x)} = 2 + 4(x-1) + \frac{6}{2}(x-1)^2 + \frac{6}{6}(x-1)^3 + \underbrace{\frac{0}{4!}(x-1)^4 + \dots}_{0 \text{ after this}}$$

$$\boxed{x^3 + x = 2 + 4(x-1) + 3(x-1)^2 + (x-1)^3}$$

HW 12 #1

Find the first four non-zero terms of a power series solution to

$$\boxed{\begin{gathered} y'' - (x+1)y' + x^2 y = 0 \\ y'(0) = 1, \quad y(0) = 1 \end{gathered}}$$

Whats the radius of convergence?

$x_0 = 0$ in this problem.

Coefficients

$$\left.\begin{aligned} -(x+1) &= -1 - x + 0x^2 + 0x^3 + \dots \\ x^2 &= 0 + 0x + 1x^2 + 0x^3 + \dots \\ 0 &= 0 + 0x + 0x^2 + 0x^3 + \dots \end{aligned}\right]$$

these are polynomials so their radius of convergence is $r = \infty$

The solution we find will have radius of convergence $r = \infty$

Find the answer!

Answer looks like

$$y(x) = y(0) + y'(0)(x-0) + \frac{y''(0)}{2!}(x-0)^2 + \frac{y'''(0)}{3!}(x-0)^3 + \cdots$$

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \cdots$$

$$\boxed{\text{Given: } y(0)=1,\ y'(0)=1}$$

Know: $y'' - (x+1)y' + x^2 y = 0$

$$y'' = (x+1)y' - x^2 y$$

$$y''(0) = (0+1)[y'(0)] - (0)^2[y(0)]$$

$$= 1 \cdot 1 - 0 \cdot 1 = 1$$

So, $y''(0) = 1$

$$y'' = (x+1)y' - x^2 y \leftarrow \text{Differentiate}$$

$$y''' = (1)y' + (x+1)y'' - 2xy - x^2 y'$$

$$y'''(0) = [y'(0)] + (0+1)[y''(0)] - 2(0)[y(0)] - (0)^2[y'(0)]$$

$$= 1 + 1\cdot 1 - 0 - 0 = 2$$

$$\boxed{y'''(0) = 2}$$

Thus,

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \dots$$

$$y(x) = 1 + 1\cdot x + \frac{1}{2!}x^2 + \frac{2}{3!}x^3 + \dots$$

$$y(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots$$