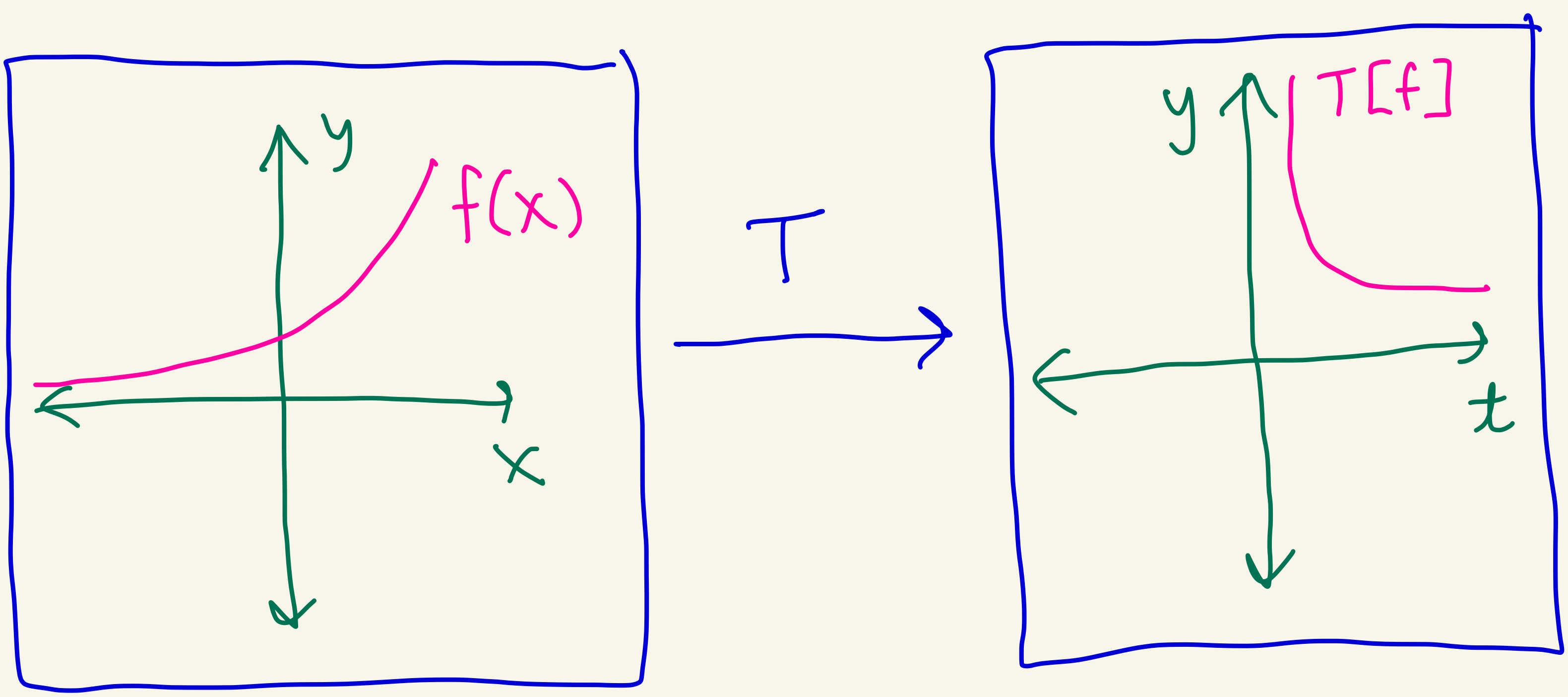
Math 2150-02

4/30/25

In mathematics an integral transform is a function T that takes a function f and transforms it into $T[f]$ using the formula:

$$T[f](t) = \int_{x_1}^{x_2} \underbrace{K(x,t)}_{\text{called the kernel of the transform}} f(x)\,dx$$

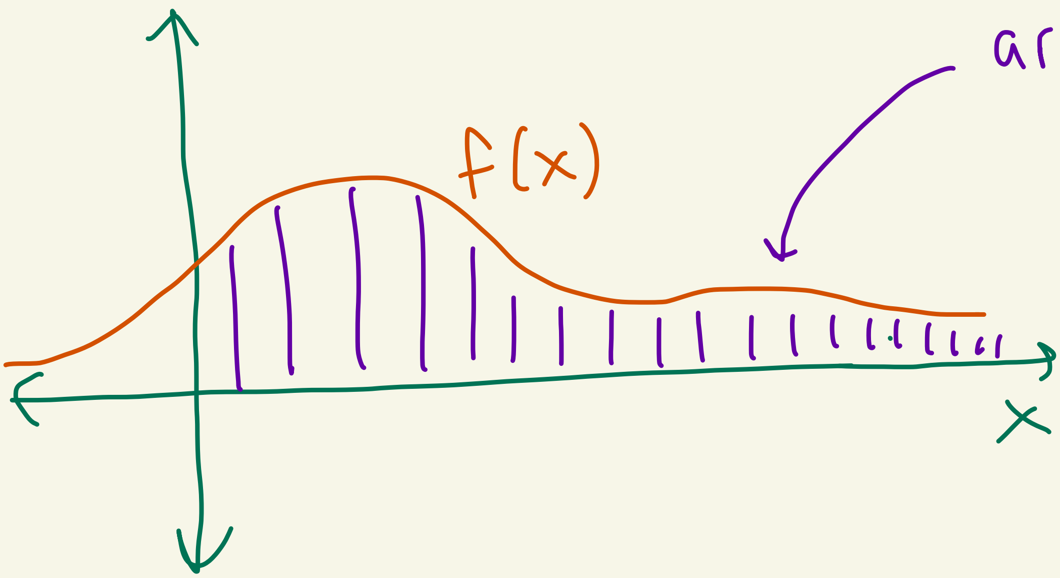


Examples

Name	Kernel
Fourier transform	$e^{-2\pi i t x}$
Laplace transform	e^{-tx}
Mellin transform	x^{t-1}
$\vdots$	$\vdots$

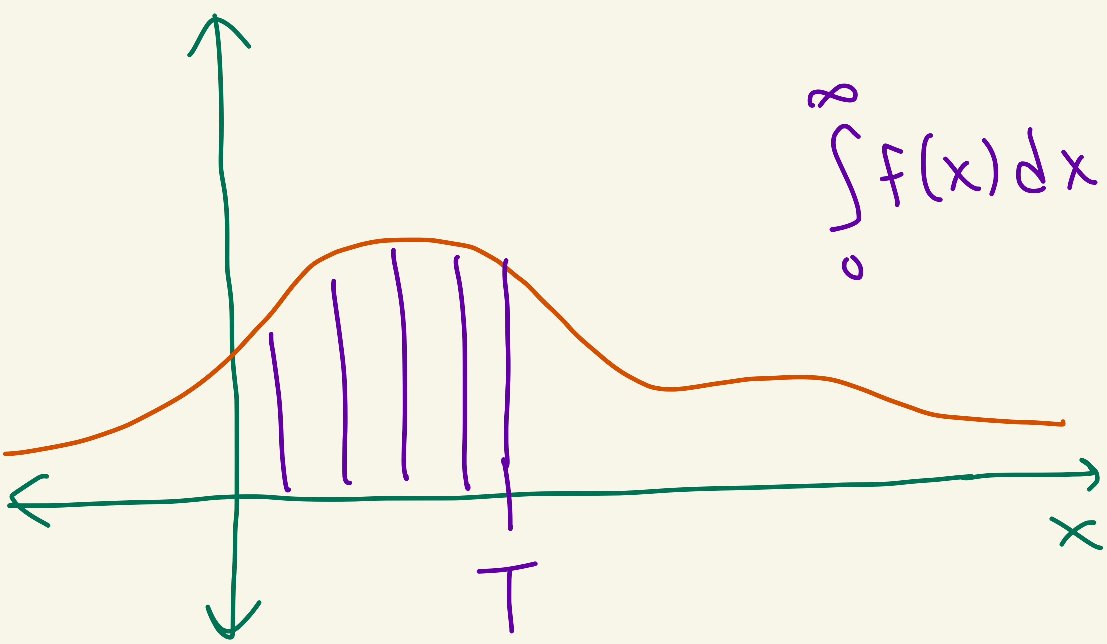
For the Laplace transform we need improper integrals.

Recall:



$$\text{area} = \int_0^\infty f(x)\,dx$$

which is defined as



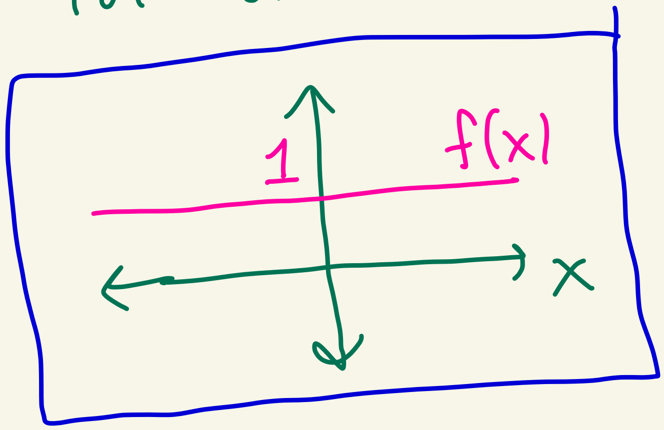
$$\int_0^\infty f(x)\,dx = \lim_{T \to \infty} \int_0^T f(x)\,dx$$

<u>Def:</u> Given a function $f(x)$ defined for $0 \le x < \infty$, let the <u>Laplace transform</u> of f be:

$$\mathcal{L}[f] = \int_0^\infty \underbrace{e^{-tx}}_{\boxed{\text{Kernel}}} f(x)\,dx$$

<u>Note:</u> $\mathcal{L}[f]$ is a function of t. So we sometimes write $\mathcal{L}[f](t)$.

Ex: Let $f(x) = 1$ for all x.



Then,

$$\mathcal{L}[f] = \int_0^\infty e^{-tx} f(x)\,dx$$

$$= \int_0^\infty e^{-tx} \cdot 1\,dx = \boxed{\int_0^\infty e^{-tx}\,dx}$$

Let's try various t values.

$$\mathcal{L}[f](2) = \int_0^\infty e^{-2x}\,dx$$

$\boxed{t=2}$

$$= \lim_{T\to\infty} \int_0^T e^{-2x}\,dx$$

$$= \lim_{T\to\infty} \left[-\frac{1}{2} e^{-2x} \Big|_0^T \right]$$

$$= \lim_{T\to\infty} \left[-\frac{1}{2} e^{-2T} + \frac{1}{2} e^{-2(0)} \right]$$

$$= \lim_{T\to\infty} \left[-\frac{1}{2} \cdot \frac{1}{e^{2T}} + \frac{1}{2} \right]$$

$$= \left[0 + \frac{1}{2} \right] = \frac{1}{2}$$

Let's try $t = -1$.

We get

$$\mathcal{L}[f](-1) = \int_0^\infty e^{-(-1)x}\, dx$$

$$= \lim_{T\to\infty} \int_0^T e^x\, dx$$

$$= \lim_{T\to\infty} \left[e^T - e^0 \right] = \infty$$

So, $\mathcal{L}[f](-1)$ is undefined.

It turns out that $\mathcal{L}[f](t)$ when $f(x)=1$ is undefined when $t\leq 0$.

But if $t>0$ then we get

$$\mathcal{L}[f](t) = \int_0^\infty e^{-tx}\,dx$$

$$= \lim_{T\to\infty} \int_0^T e^{-tx}\,dx$$

$$= \lim_{T\to\infty} \left[-\frac{1}{t} e^{-tx} \Big|_{x=0}^{T} \right]$$

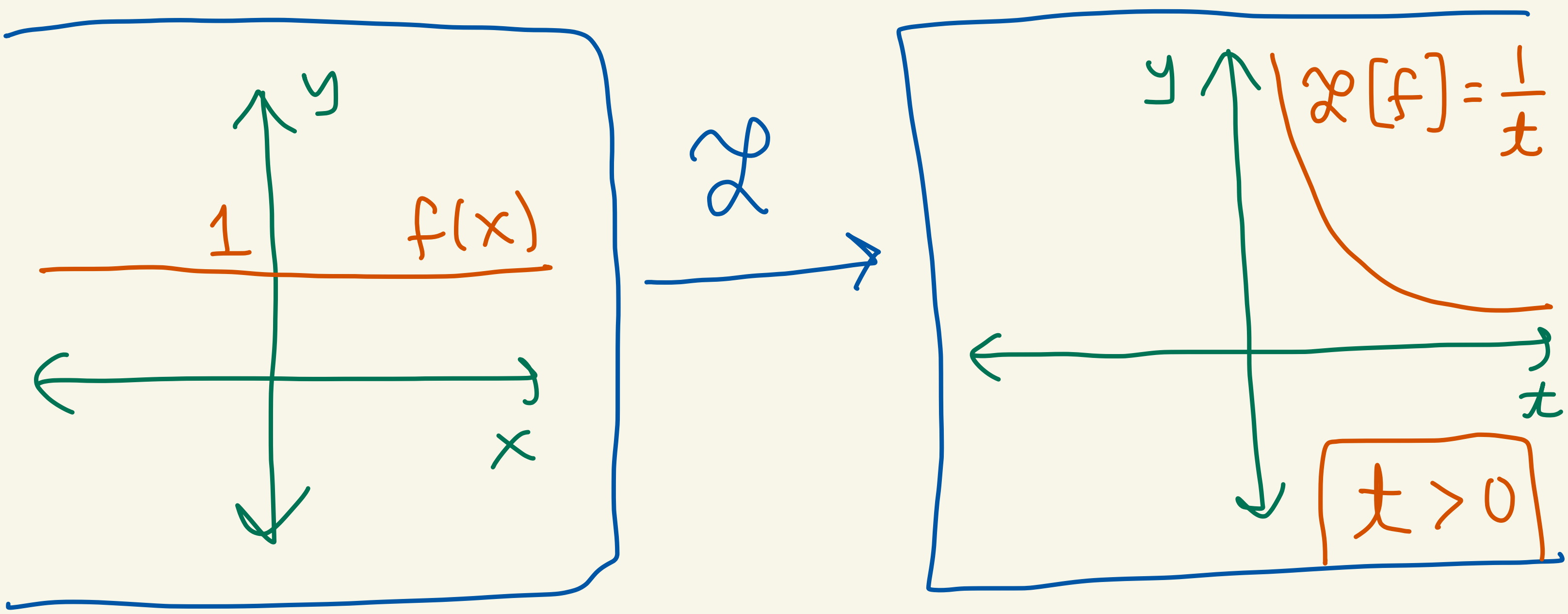
$$= \lim_{T\to\infty} \left[-\frac{1}{t} e^{-tT} + \frac{1}{t} e^{0} \right]$$

$$= \lim_{T\to\infty} \left[-\frac{1}{t} \cdot \frac{1}{\underbrace{e^{tT}}} + \frac{1}{t} \right]$$

as $T \to \infty$ because $t > 0$

0

$$= 0 + \frac{1}{t} = \frac{1}{t}$$



Ex: Let $f(x) = e^{ax}$ where a is any constant.

If $t > a$, then

$$\mathcal{L}[f] = \int_0^\infty e^{-tx} f(x)\,dx$$

$$= \int_0^\infty e^{-tx} e^{ax}\,dx$$

$$= \lim_{T \to \infty} \int_0^T e^{(a-t)x}\,dx$$

$$= \lim_{T \to \infty} \left[\frac{1}{(a-t)} e^{(a-t)x} \Big|_{x=0}^{T} \right]$$

$$= \lim_{T \to \infty} \left[\frac{1}{(a-t)} \underbrace{e^{(a-t)T}} - \frac{1}{(a-t)} e^{0} \right]$$

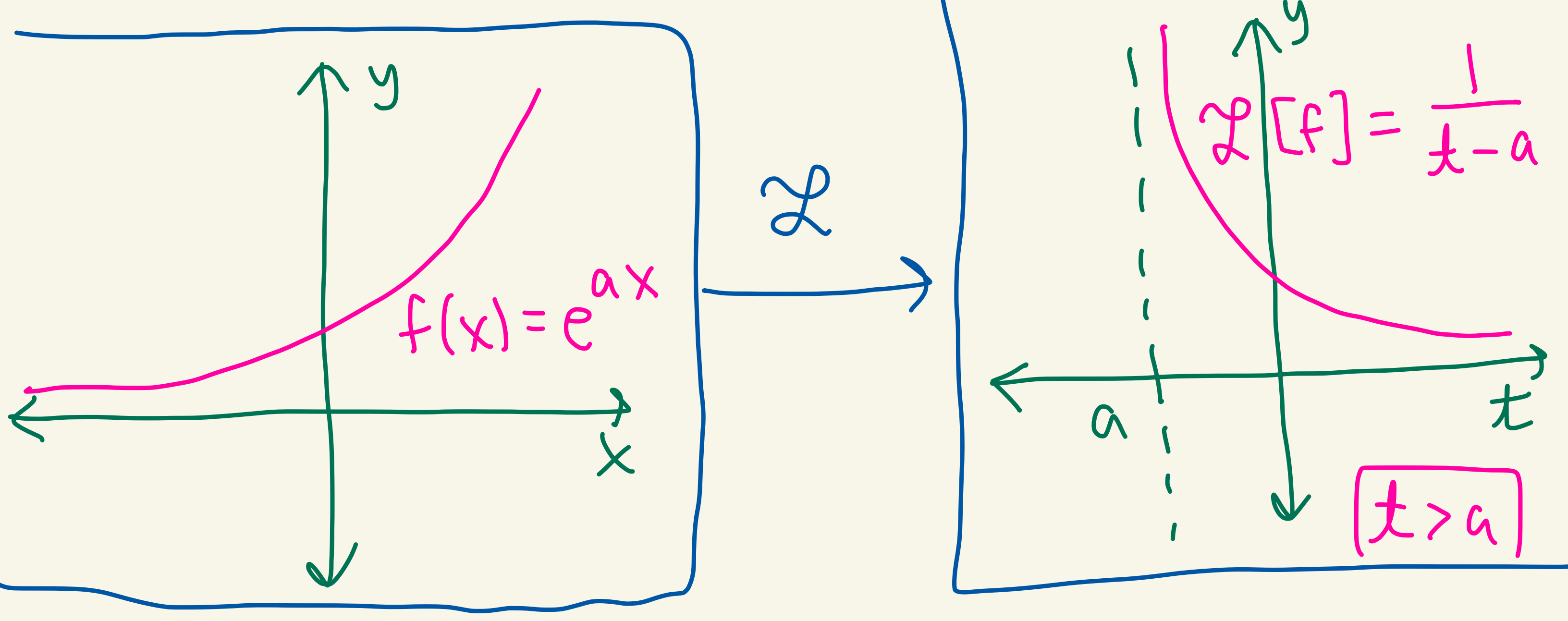
$t > a$
$0 > a - t$

0 as $T \to \infty$

$$= \left[0 - \frac{1}{a-t} \cdot 1 \right]$$

$$= \frac{1}{t-a}$$

The above integral would diverge if $t \leq a$.



Some Laplace Transforms

① $\mathcal{L}[1] = \dfrac{1}{t}$ where $t > 0$

② $\mathcal{L}[x^n] = \dfrac{n!}{t^{n+1}}$ where $t > 0$
$n = 1, 2, 3, \ldots$

③ $\mathcal{L}[e^{ax}] = \dfrac{1}{t-a}$ where $t > a$

④ $\mathcal{L}[\sin(kx)] = \dfrac{k}{t^2+k^2}$ where $t > 0$

⑤ $\mathcal{L}[\cos(kx)] = \dfrac{t}{t^2+k^2}$ where $t > 0$

Theorem:

Suppose $\mathcal{L}[f]$ and $\mathcal{L}[g]$ both exist for $t > t_0$.

Then,

$$\mathcal{L}[c_1 f + c_2 g] = c_1 \mathcal{L}[f] + c_2 \mathcal{L}[g]$$

when $t > t_0$.

Here c_1, c_2 are constants.

Furthmore, if f, f', f'' are continuous on $[0, \infty)$ and their Laplace transforms exist then

$$\mathcal{L}[f'] = t \cdot \mathcal{L}[f] - f(0)$$

$$\mathcal{L}[f''] = t^2 \mathcal{L}[f] - t f(0) - f'(0)$$