$$\boxed{\begin{array}{l}\text{Math 2150-02}\\ \text{4/7/25}\end{array}}$$

Topic 11 - Review of power series

<u>Def:</u> An <u>infinite sum</u> is a sum of the form

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + a_3 + \cdots$$

[Here we started at $n=0$, but you can start at any number]

We say that the sum <u>converges</u> to a number S if

$$\underbrace{\lim_{N\to\infty} \sum_{n=0}^{N} a_n}_{} = S$$

$$\lim_{N\to\infty} [\underbrace{a_0 + a_1 + \cdots + a_N}_{\text{called a partial sum}}]$$

If there is no such S, then the sum <u>diverges</u>.

<u>Ex:</u> Consider

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots$$

Let's look at the partial sums:

N	$\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^N}$
1	$\frac{1}{2} = 0.5$
2	$\frac{1}{2} + \frac{1}{2^2} = 0.75$
3	$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} = 0.875$
4	$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} = 0.9375$
5	0.96875

⋮	⋮
50	$0.\underbrace{9999999\ldots99}_{15\ 9\text{'s}}11182\ldots$
⋮	⋮
100	$0.\underbrace{999\ldots\ldots999}_{30\ 9\text{'s}}11139\ldots$

In Calculus 2120 you show that

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1$$

Def: A power series is an infinite sum of the form

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n$$

$$= a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots$$

where the a_i and x_0 are constants, and x is a variable. We say that the power series is centered at x_0.

Ex: (Geometric series)

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

$$\underbrace{1}_{a_0} + \underbrace{1}_{a_1}\cdot(x-\underbrace{0}_{x_0}) + \underbrace{1}_{a_2}\cdot(x-\underbrace{0}_{x_0})^2 + \underbrace{1}_{a_3}\cdot(x-\underbrace{0}_{x_0})^3 + \cdots$$

(center is $x_0 = 0$)

From Calculus:

- If $-1 < x < 1$, then
$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$
- The sum diverges for other values of x.

For example,

$$\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n = 1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots$$

$$\boxed{S}$$

$$= \frac{1}{1-\left(\frac{1}{3}\right)} = \frac{1}{2/3} = \boxed{\frac{3}{2}}$$

$x = 1/3$
$-1 < x < 1$

And

$$\sum_{n=0}^{\infty} \left(\frac{10}{3}\right)^n = 1 + \frac{10}{3} + \left(\frac{10}{3}\right)^2 + \left(\frac{10}{3}\right)^3 + \cdots$$

diverges since $x = \frac{10}{3}$ does not satisfy $-1 < x < 1$.

You can think of the power series as a function of x.

For example, let

$$f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

Then

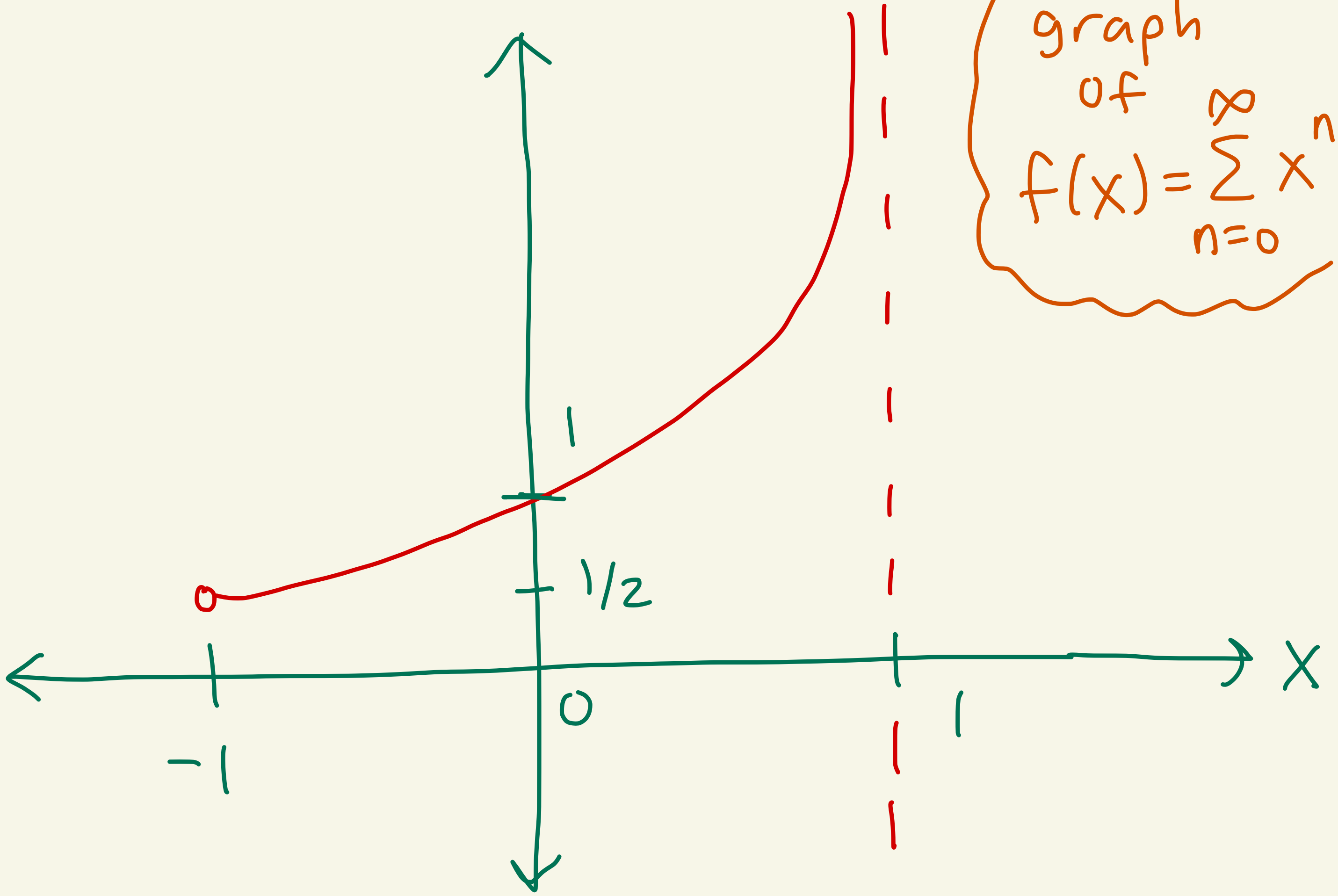
$$f(0) = 1 + 0 + 0^2 + 0^3 + \cdots = 1$$

$$f\left(-\frac{2}{3}\right) = 1 + \left(-\frac{2}{3}\right) + \left(-\frac{2}{3}\right)^2 + \cdots = \frac{1}{1-\left(-2/3\right)}$$

$= 3/5$

$$\boxed{\begin{array}{c} x = -2/3 \\ -1 < x < 1 \end{array}}$$

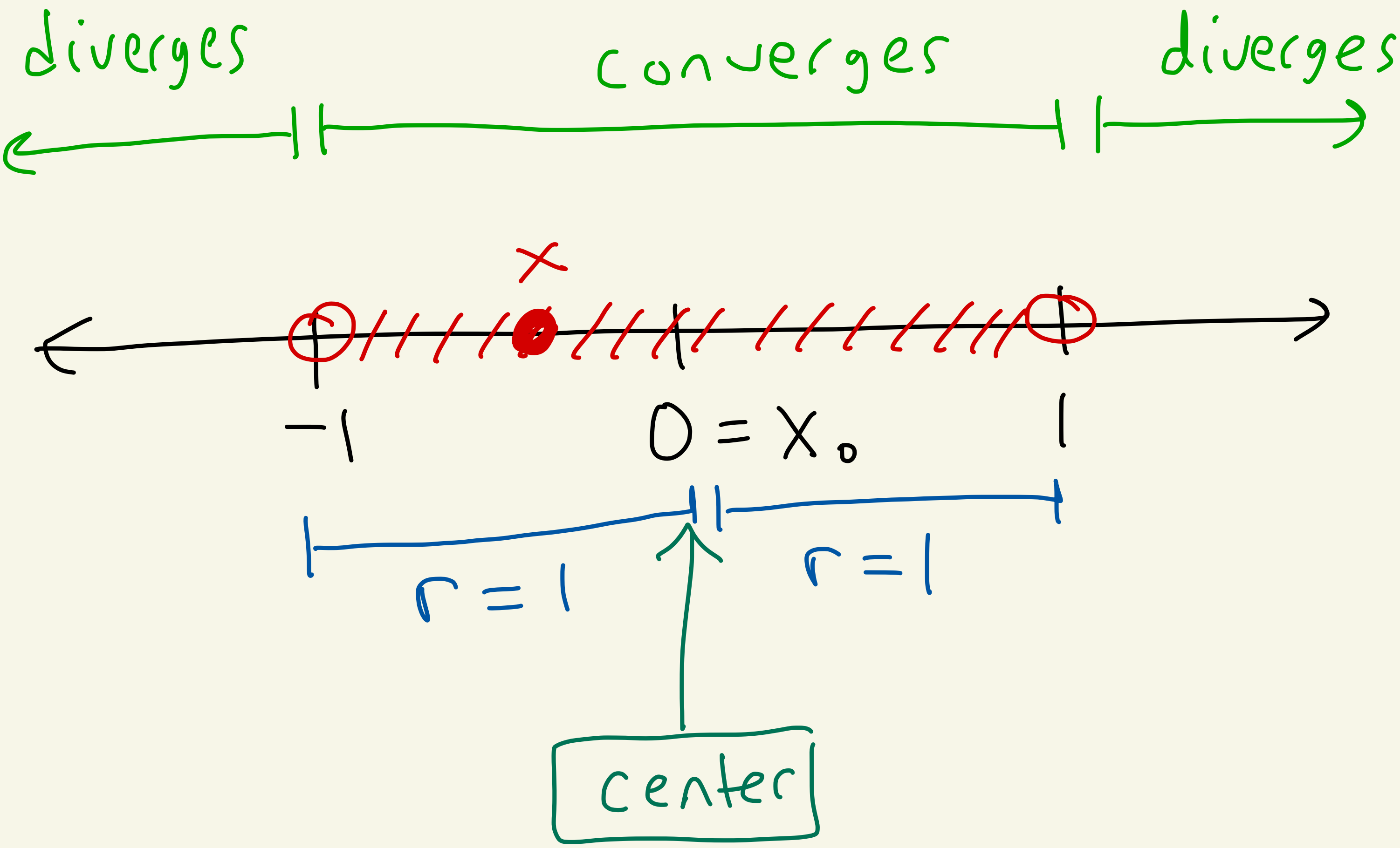
$$f(100) = 1 + 100 + 100^2 + 100^3 + \dots = \infty$$



graph of $f(x) = \sum_{n=0}^{\infty} x^n$

$f(x)$ is only defined for $-1 < x < 1$ and it equals $\frac{1}{1-x}$ there.

This function converges for these x's:



$r=1$ is called the radius of convergence.

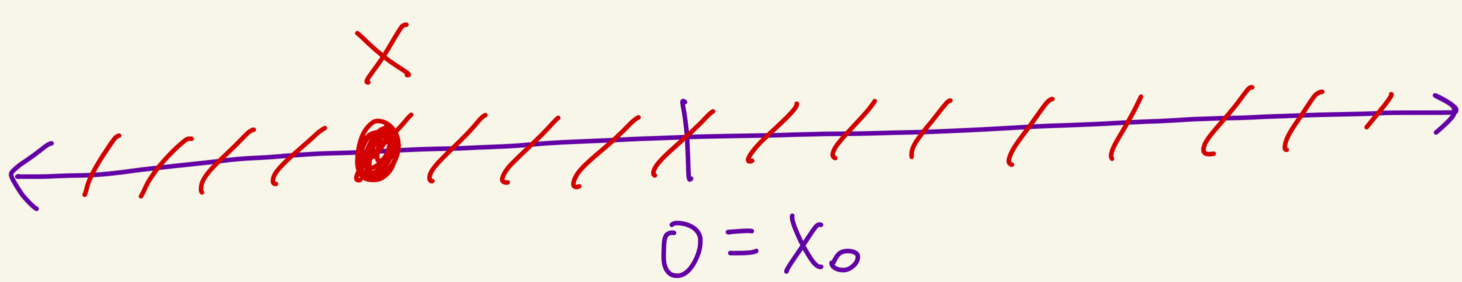
Ex: Recall from Math 2120, (Calc II)

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{1}{1!}x^1 + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$$

$$= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots$$

(center is $x_0 = 0$)

This series converges for all x.



The radius of convergence is $r = \infty$