Math 4460

4/14/25

# Application – Pythagorean Triples

Consider the equation

$$\boxed{x^2 + y^2 = z^2}$$

We will find all integer solutions to this equation.

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<u>Ex:</u> $x = 3, y = 4, z = 5$ is a solution. Let $k \in \mathbb{Z}$. Then, $x = 3k, y = 4k, z = 5k$ is also a solution because

$$3^2 + 4^2 = 5^2$$

$$k^2\left[3^2 + 4^2\right] = k^2\left[5^2\right]$$

$$3^2k^2 + 4^2k^2 = 5^2k^2$$

$$(3k)^2 + (4k)^2 = (5k)^2$$

So we can get an infinite # of solutions by varying $k$.

| $k$ | $x = 3k$ | $y = 4k$ | $z = 5k$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 3 | 4 | 5 |
| −1 | −3 | −4 | −5 |
| 2 | 6 | 8 | 10 |
| −10 | −30 | −40 | −50 |

SOME EXAMPLES

Another way to get more solutions from a single solution is vary the signs.

Ex:

Start with $x=3, y=4, z=5$.

Get these 8 solutions:

$$(x,y,z) = (3,4,5), (-3,4,5)$$
$$(-3,-4,5), (-3,4,-5)$$
$$(-3,-4,-5), (3,-4,5)$$
$$(3,-4,-5), (3,4,-5)$$

Another way to get solutions is set one or all variables to be zero.

Some solutions to $x^2+y^2=z^2$ are

$$x=2, y=0, z=2$$

$$x=0, y=-10, z=10$$

and so on.

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**Def:** We call $(x,y,z)$ a **Pythagorean triple** if

① $x, y, z \in \mathbb{Z}$

② $x^2 + y^2 = z^2$

③ $(x,y,z) \neq (0,0,0)$ ] not all zero

If in addition, $x > 0, y > 0, z > 0$ then we call $(x,y,z)$ a **positive Pythagorean triple**.

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**Ex:**

$(3,-4,5)$
$(0,3,3)$ ] Pythagorean triples

$(3,4,5)$ ] positive Pythagorean triple

Ex: $(x,y,z) = (25, 60, -65)$ is a Pythogean triple because

$$x^2 + y^2 = 25^2 + 60^2 = 625 + 3600$$
$$= 4225$$
$$= (-65)^2 = z^2$$

So, $x^2 + y^2 = z^2$

Let $d = \gcd(25, 60, -65)$

Then, $d = 5$.

And,

$$(x,y,z) = (25, 60, -65)$$
$$= (5 \cdot 5, 5 \cdot 12, -5 \cdot 13)$$
$$= (d \cdot 5, d \cdot 12, -d \cdot 13)$$

Also, $(5,12,13)$ is a positive Pythagorean triple with $\gcd(5,12,13)=1$.

We can "make" $(25,60,-65)$ from the positive Pythagorean triple $(5,12,13)$ like this:

$$(5,12,13) \xrightarrow{\times 5} (25,60,65) \xrightarrow{\pm 1} (25,60,-65)$$

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<u>Def</u>: Let $(x,y,z)$ be a Pythagorean triple. We say that $(x,y,z)$ is <u>primitive</u> if $\gcd(x,y,z)=1$.

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**Theorem:** Any Pythagorean triple is of the form $(\pm da, \pm db, \pm dc)$ where $(a,b,c)$ is a primitive Pythagorean triple and $d > 0$. $a \geq 0, b \geq 0, c \geq 0$

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**Ex:** $(9,-12,-15) \leftarrow$ Pythagorean triple

$$(9,-12,-15) = (3\cdot 3, -3\cdot 4, -3\cdot 5)$$

$$d=3, \quad \underbrace{(a,b,c)=(3,4,5)}_{\substack{\text{primitive} \\ \gcd(3,4,5)=1}}$$

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**Ex:** $(0,4,-4) \leftarrow$ Pythagorean triple

$(0,4,-4) = (4\cdot 0, 4\cdot 1, -4\cdot 1)$

$d=4$, $\underbrace{(a,b,c)=(0,1,1)}_{\text{primitive}}$

$\gcd(0,1,1)=1$

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Proof of theorem:

Let $(x,y,z)$ be a Pythagorean triple.

Then, $x^2+y^2=z^2$ and $(x,y,z)\neq(0,0,0)$

and $x,y,z\in\mathbb{Z}$

Let $d=\gcd(x,y,z)$.

From class, $\gcd\left(\frac{x}{d},\frac{y}{d},\frac{z}{d}\right)=1$.

Set

$a=\left|\frac{x}{d}\right|$, $b=\left|\frac{y}{d}\right|$, $c=\left|\frac{z}{d}\right|$

Then,

$$(x,y,z) = (\pm da, \pm db, \pm dc)$$

where +/- depends on the signs of $a, b, c$.

We see $a \geq 0$, $b \geq 0$, $c \geq 0$.

Since $d \mid x$, $d \mid y$, $d \mid z$ we know $a, b, c$ are integers.

And $\gcd(a,b,c) = 1$.

Furthermore,

$$a^2 + b^2 = \left|\frac{x}{d}\right|^2 + \left|\frac{y}{d}\right|^2$$

$$= \left(\frac{x}{d}\right)^2 + \left(\frac{y}{d}\right)^2$$

$$= \frac{1}{d^2}(x^2 + y^2)$$

$$= \frac{1}{d^2}(z^2)$$

because $(x,y,z)$ is a triple.

$$= \left(\frac{z}{d}\right)^2$$

$$= \left|\frac{z}{d}\right|^2 = c^2$$

Thus, $(a, b, c)$ is a Pythagorean triple and its primitive $\blacksquare$ —

Summary: There are three kinds of Pythagorean triples.

① $(x, 0, \pm x)$ ← Ex: $(3, 0, -3)$
$3^2 + 0^2 = (-3)^2$

② $(0, y, \pm y)$ ← Ex: $(0, 7, 7)$
$0^2 + 7^2 = 7^2$

③ the ones that are multiples of positive primitive Pythagorean triples with possible sign adjustments

$(5, 12, 13)$ positive primitive Pythagorean triple $\xrightarrow{\times 2}$ $(10, 24, 26)$ $\xrightarrow{\text{sign adjustment}}$ $(10, -24, -26)$

Thus, we just need a formula for the positive, primitive Pythagorean triples.