Math 5800

8/25/21

Theorem: Let A and B be sets with $A \neq \emptyset$, $B \neq \emptyset$. If $A \subseteq B$ and B is countable, then A is countable.

proof: If B is finite, then since $A \subseteq B$ we know A is finite. Thus, A is countable.

Now suppose B is countably infinite. Then, we can enumerate the elements of B in a sequence with no repeats, that is

$$B = \{ b_1, b_2, b_3, b_4, \ldots \}$$

Since $A \subseteq B$ we can go through B's sequence and pick out the elements of A and skip the others and you'll get a sub-sequence enumerating A, that is

$$A = \{ b_{i_1}, b_{i_2}, b_{i_3}, \ldots \}$$

where $i_1 < i_2 < i_3 < \ldots$

Thus, A is countable. $\blacksquare$

Def: Let S be a set, with $S \neq \emptyset$.
If S is not <u>countable</u> then we say that S is <u>uncountable</u>.

Ex: Some uncountable numbers

$\mathbb{R}$

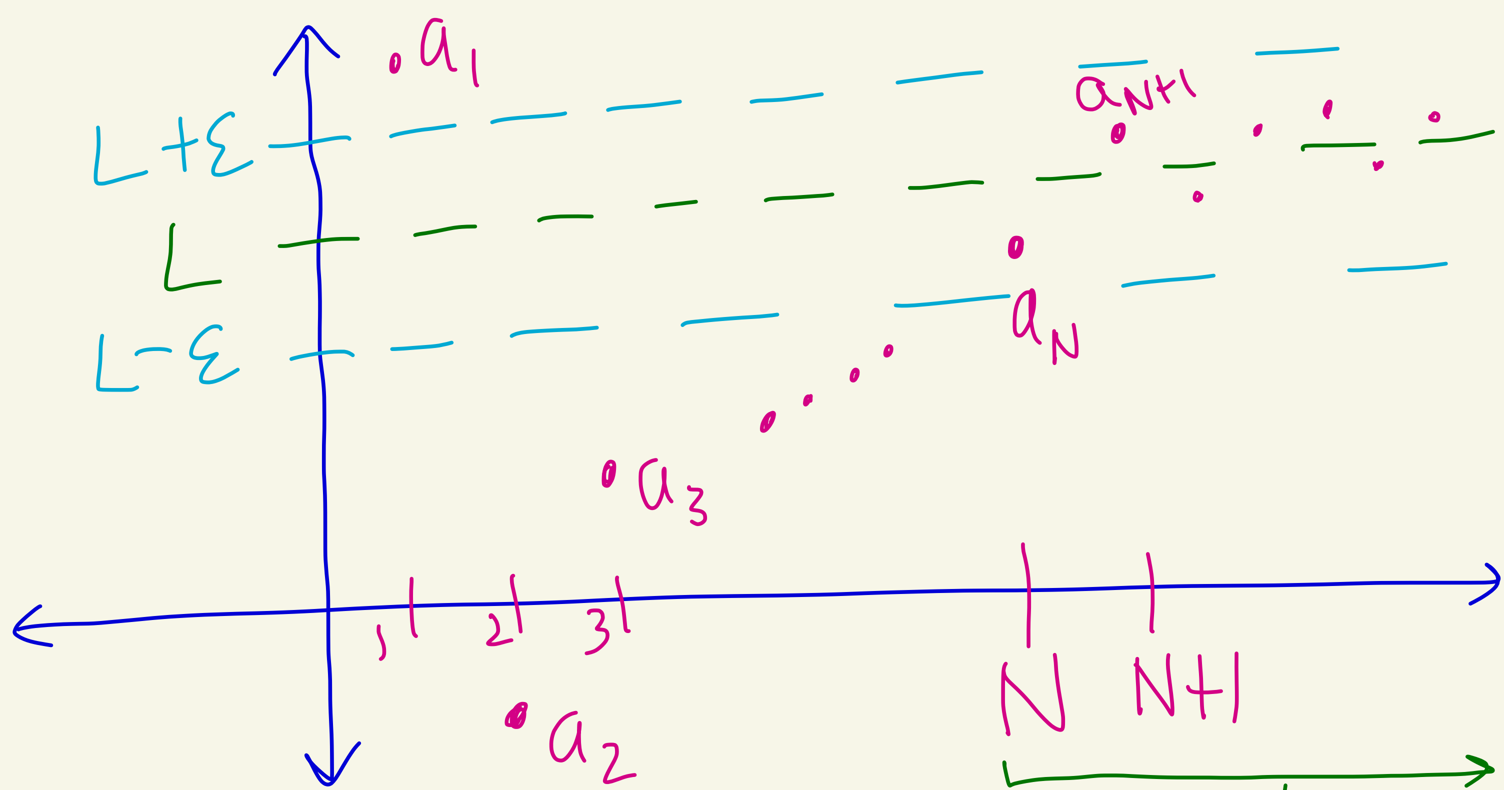
$\mathbb{R} - \mathbb{Q}$ ← irrationals

$[0, 1]$

$(10, 10.5)$

Topic 2 – 4650 Review

Def: Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. Let $L \in \mathbb{R}$. We say that $\lim_{n\to\infty} a_n = L$ if for every $\varepsilon > 0$ there exists $N > 0$ where if $n \geq N$ then $|a_n - L| < \varepsilon$

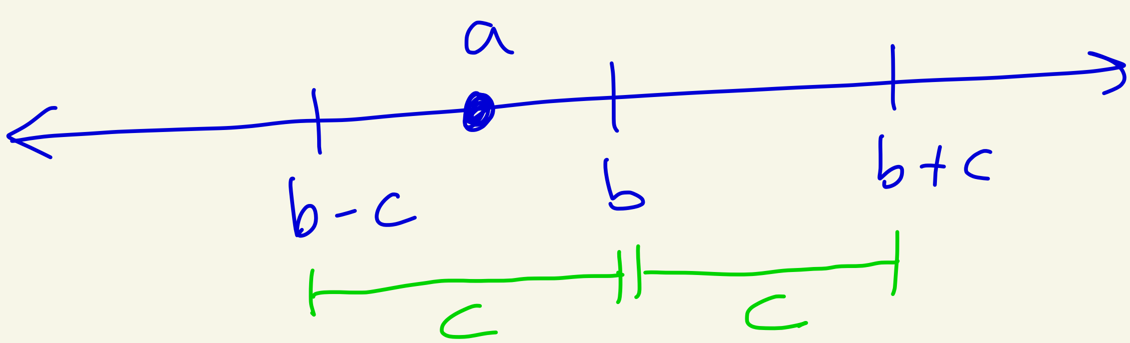


If such an L exists then we say that the sequence $(a_n)_{n=1}^{\infty}$ <u>converges</u>. If no such L exists then we say that the sequence <u>diverges</u>

<u>Recall:</u> Let $a,b,c \in \mathbb{R}$ with $c>0$.

Then,

$$\boxed{|a-b|<c} \quad \text{iff} \quad \boxed{b-c<a<b+c}$$



So,

$$\boxed{|a_n-L|<\varepsilon} \quad \text{iff} \quad \boxed{L-\varepsilon<a_n<L+\varepsilon}$$

<u>Theorem:</u> Let $(a_n)_{n=1}^{\infty}$ converge to A and $(b_n)_{n=1}^{\infty}$ converge to B.

Let $\alpha, \beta \in \mathbb{R}$.

Then,

① $\lim\limits_{n\to\infty} \left[\alpha a_n + \beta b_n\right] = \alpha A + \beta B$

② $\lim\limits_{n\to\infty} \left[a_n b_n\right] = AB$

③ $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{A}{B}$, if $B \neq 0$ and $b_n \neq 0$ for all n

<u>Def:</u> Given a sequence of real numbers

$a_1, a_2, a_3, a_4, a_5, \ldots$

define

$$
\begin{aligned}
S_1 &= a_1 \\
S_2 &= a_1 + a_2 \\
S_3 &= a_1 + a_2 + a_3 \\
&\vdots \\
S_k &= \sum_{n=1}^{k} a_n = a_1 + a_2 + \cdots + a_k
\end{aligned}
$$

S_k is called the k-th partial sum

If $\lim_{k\to\infty} S_k = \lim_{k\to\infty} \sum_{n=1}^{k} a_k$ converges to a real number L, then we say that $\sum_{n=1}^{\infty} a_n$ <u>converges</u> to L

and write $\sum_{n=1}^{\infty} a_n = L$

Ex: Show that

$$\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = 2$$

We have

$$S_k = \sum_{n=0}^{k}\frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^k}$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \dots + \left(\frac{1}{2}\right)^k$$

$x = \frac{1}{2}$

$$= \frac{1-\left(\frac{1}{2}\right)^{k+1}}{1-\frac{1}{2}}$$

If $x \in \mathbb{R}$ and $x \neq 1$ then

$$1 + x + x^2 + \dots + x^m = \frac{1-x^{m+1}}{1-x}$$

Geometric series

So,

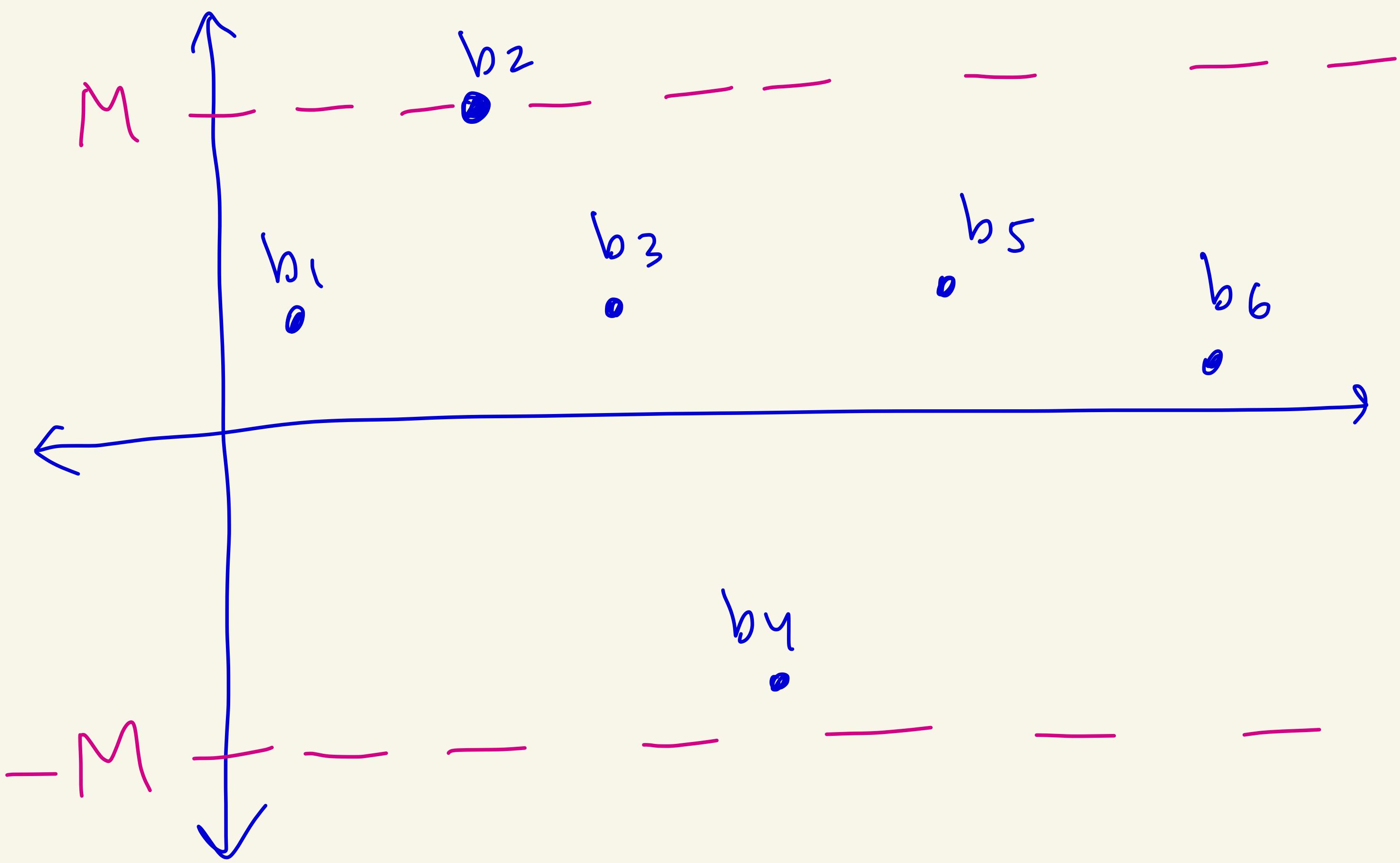
$$\lim_{k\to\infty} S_k = \lim_{k\to\infty}\left[\frac{1-\left(\frac{1}{2}\right)^{k+1}}{1-\frac{1}{2}}\right]$$

$$= \left[\frac{1-0}{1-\frac{1}{2}}\right]$$

$\lim r^n = 0$ if $-1 < r < 1$

$$= 2$$

Def: Let $(b_n)_{n=1}^{\infty}$ be a sequence of real numbers. We say that $(b_n)_{n=1}^{\infty}$ is **bounded** if there exists $M > 0$ where $\underbrace{|b_n| \leq M}_{-M \leq b_n \leq M}$ for all n.

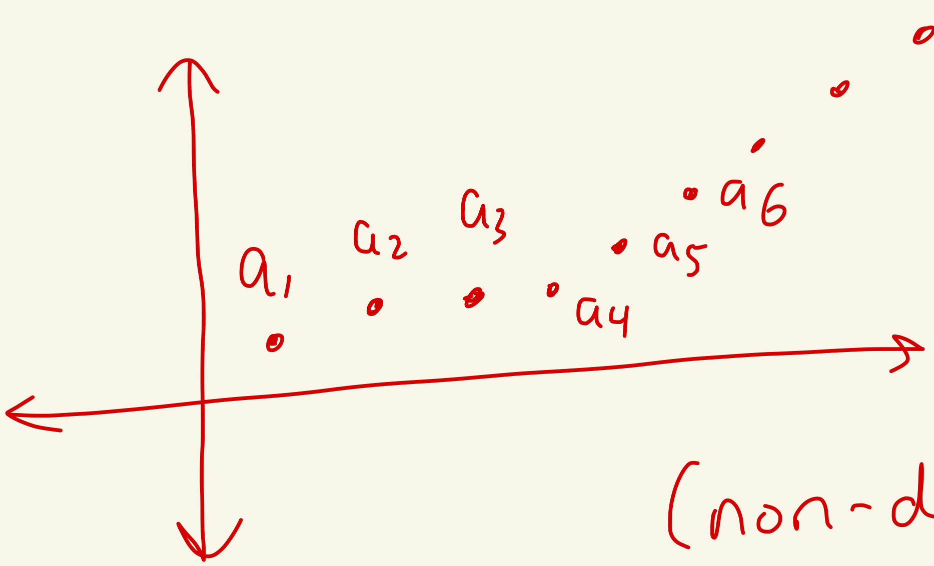


Theorem: Let $(b_n)_{n=1}^{\infty}$ be a sequence of real numbers. If $(b_n)_{n=1}^{\infty}$ converges, then $(b_n)_{n=1}^{\infty}$ is bounded.

proof: HW. $\blacksquare$

Def: Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. We say that $(a_n)_{n=1}^{\infty}$ is <u>non-decreasing</u> if

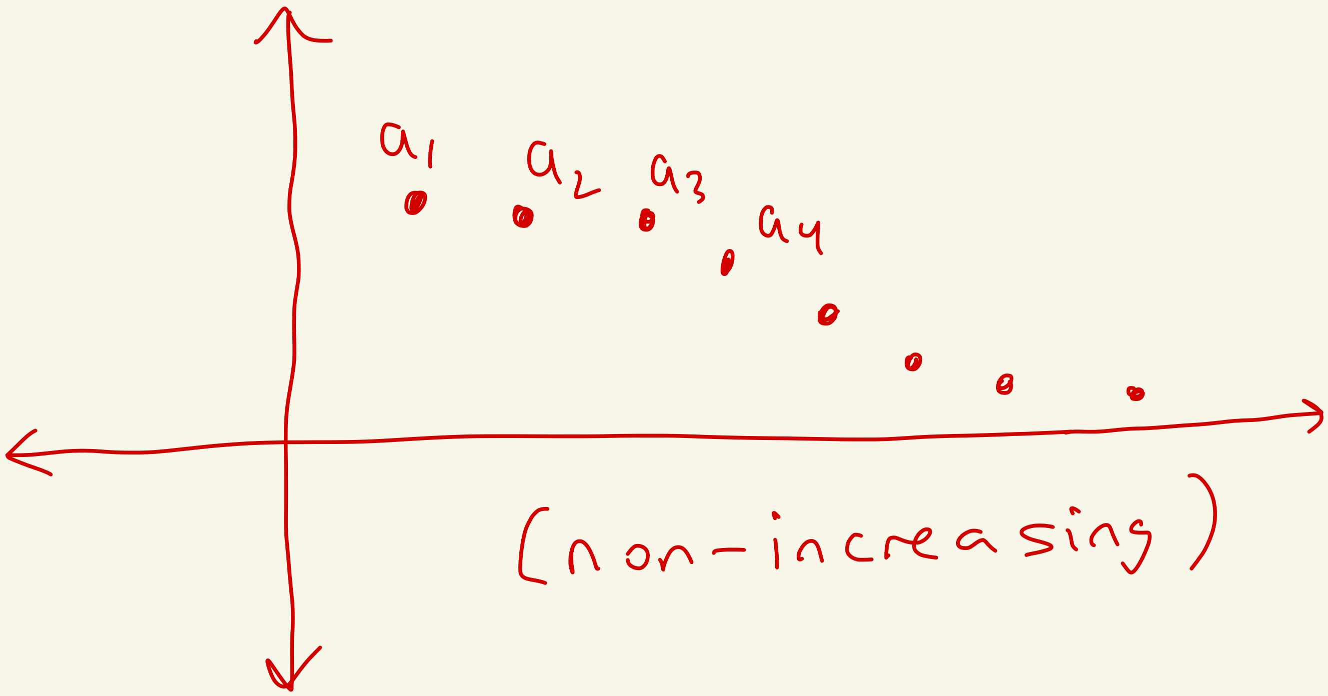
$$a_n \leq a_{n+1} \text{ for all } n.$$



(non-decreasing)

We say that $(a_n)_{n=1}^{\infty}$ is **non-increasing** if

$a_{n+1} \leq a_n$ for all n.

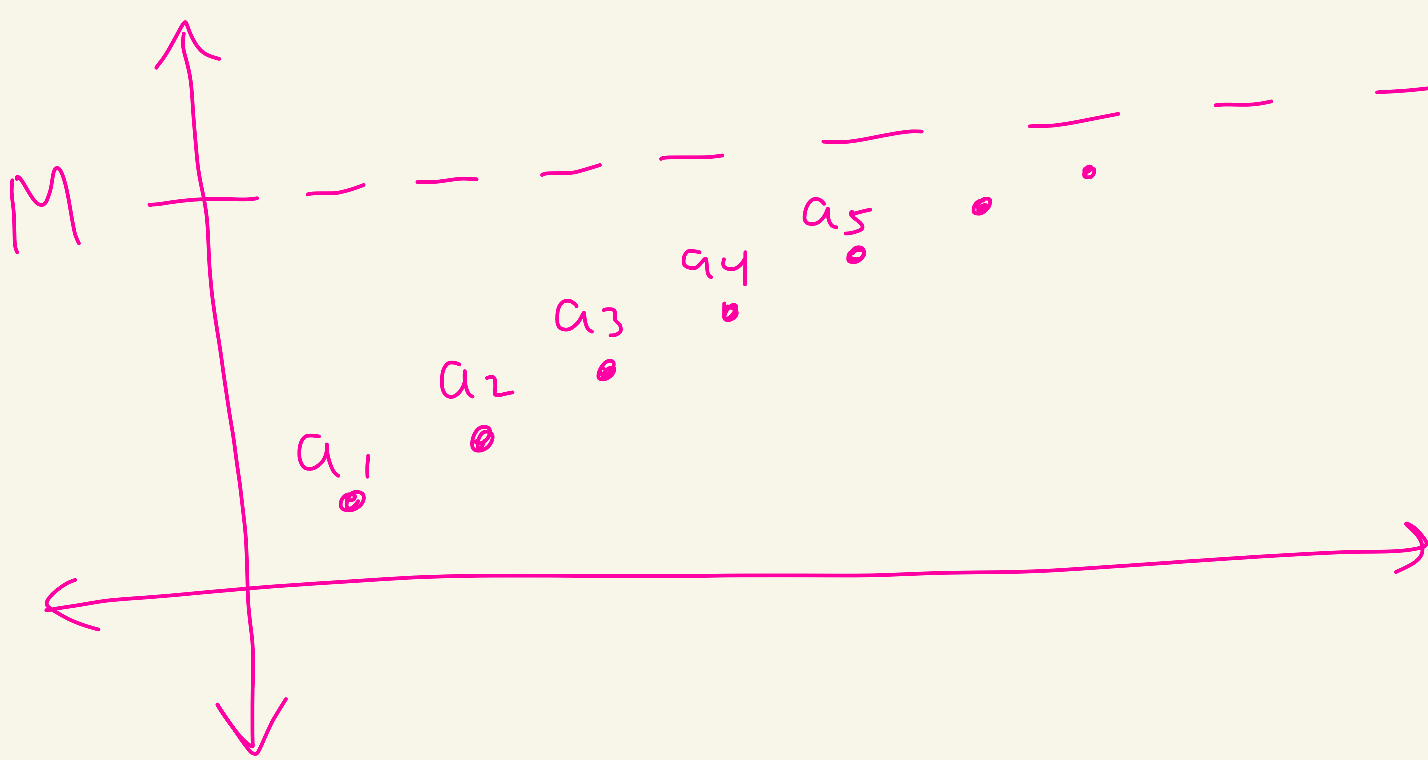


(non-increasing)

Theorem: (Monotone Convergence Theorem)

① If $(a_n)_{n=1}^{\infty}$ is a non-decreasing sequence that is bounded from above, then $(a_n)_{n=1}^{\infty}$ converges.

[Bounded from above means there exists $M \in \mathbb{R}$ where $a_n \leq M$ for all n.]



② If $(a_n)_{n=1}^{\infty}$ is a non-increasing sequence that is bounded from below, then $(a_n)_{n=1}^{\infty}$ converges.

Bounded from below means there exists $M \in \mathbb{R}$ where $M \leq a_n$ for all n.