California State University – Los Angeles

Department of Mathematics

Master's Degree Comprehensive Examination

Analysis Fall 2024

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Do at least two (2) problems from Section 1 below, and at least three (3) problems from Section 2 below. All problems count equally. If you attempt more than two problems from Section 1, the best two will be used. If you attempt more than three problems from Section 2, the best three will be used. Be sure to show your work for all answers.

(1) Write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only.
(3) Begin each problem on a new page.
(4) Assemble the problems you hand in in numerical order.

Exams are graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

SECTION 1 – Do two (2) problems from this section. If you attempt all three, then the best two will be used for your grade.

Spring 2024 #1. Let $\mathbb{R}$ denote the set of real numbers, and let $\mathbb{Q}$ denote the set of rational numbers.

Define $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Prove that for all $a \in \mathbb{R}$, we have that f is not continuous at a.

SOLUTION:

There are many ways to prove this. One can be found here:

`https://en.wikipedia.org/wiki/Dirichlet_function`

Spring 2024 #2. Use the definition of limits to show that

$$\lim_{n\to\infty} \frac{1}{(n+1)^2} = 0.$$

Proof. Let $\epsilon > 0$. We must find a natural number N such that

$$\left| \frac{1}{(n+1)^2} - 0 \right| < \epsilon.$$

Note that this can be rewritten as

$$\frac{1}{(n+1)^2} < \epsilon.$$

First, we may use the Archimedean property to find $N \in \mathbb{N}$ so that

$$\frac{1}{\epsilon} < N.$$

Next, let $n \geq N$, $n \in \mathbb{N}$. Then

$$\frac{1}{\epsilon} < N \leq n < n+1 < (n+1)^2,$$

from which we conclude that

$$\frac{1}{(n+1)^2} < \epsilon$$

whenever $n \geq N$. $\square$

Spring 2024 #3. Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in $\mathbb{R}$. Show that

$$\liminf(x_n + y_n) \geq \liminf x_n + \liminf y_n.$$

Proof. Because $\{x_n\}$ and $\{y_n\}$ are bounded sequences, so is $\{x_n + y_n\}$, so all the lim infs here exist.

Let $L = \liminf(x_n + y_n)$. Let $L_x = \liminf x_n$ and $L_y = \liminf y_n$. We will show that $L \geq L_x + L_y$.

Let $\epsilon > 0$. Then there exists $M_x \in \mathbb{N}$ such that if $n \geq M_x$, then $x_n \geq L_x - \frac{1}{2}\epsilon$. Similarly, there exists $M_y \in \mathbb{N}$ such that if $n \geq M_y$, then $y_n \geq L_y - \frac{1}{2}\epsilon$.

Let $M = \max\{M_x, M_y\}$. It follows that if $n \geq M$, then

$$x_n + y_n \geq (L_x - \frac{1}{2}\epsilon) + (L_y - \frac{1}{2}\epsilon) = L_x + L_y - \epsilon.$$

Because ϵ was arbitrary, it follows that $L \geq L_x + L_y$. $\square$

SECTION 2 – Do three (3) problems from this section. If you attempt more than three, then the best three will be used for your grade.

Spring 2024 #4. Let $C([0,1])$ denote the set of continuous functions on $[0,1]$, with the L^∞ norm defined by

$$\|f\|_{L^\infty} = \sup_{x\in[0,1]} |f(x)|.$$

Let $h \in C([0,1])$. For $f \in L^2([0,1])$, define $T(f) = h(x)f(x)$.

(a) Show that T maps $L^2([0,1])$ to itself.
(b) Show that T is a bounded operator.
(c) Show that $\|T\| \le \|h\|_{L^\infty}$.

Proof. For part (a): Observe that

$$\begin{aligned}
\|T(f)\|_{L^2([0,1])} &= \left(\int_0^1 |T(f)(x)|^2 \, dx\right)^{\frac{1}{2}} \\
&= \left(\int_0^1 |h(x)f(x)|^2 \, dx\right)^{\frac{1}{2}} \\
&\le \left(\int_0^1 \left(\sup_{x\in[0,1]} |h(x)|\right)^2 |f(x)|^2 \, dx\right)^{\frac{1}{2}} \\
&= \|h\|_{L^\infty} \|f\|_{L^2}.
\end{aligned}$$

Since each norm on the right-hand side is finite, it follows that

$$T(f) \in L^2.$$

For part (b): Set $k = \|h\|_{L^\infty}$. Then the previous inequality implies that there exists a $k > 0$ such that

$$\|T(f)\|_{L^2} \le k\|f\|_{L^2}.$$

By definition, we have that T is a bounded operator on L^2.

For part (c): Let $\|f\|_{L^2} \le 1$. Then the inequality above implies that

$$\|T(f)\|_{L^2} \le \|h\|_{L^\infty},$$

and this holds for all such f. Taking the supremum, we see that

$$\|T\| = \sup\{\|T(f)\|_{L^2} : \|f\|_{L^2} \leq 1\} \leq \|h\|_{L^\infty}.$$

$\square$

Spring 2024 #5. Recall that

$$\ell^2 = \left\{ (a_1, a_2, a_3, \dots) \mid a_1, a_2, a_3, \dots \in \mathbb{C} \text{ and } \sum_{j=1}^{\infty} |a_j|^2 < \infty \right\}.$$

In other words, ℓ^2 is the set of all square-summable sequences of complex numbers. Let

$$W = \{(a_1, a_2, a_3, \dots) \mid a_1, a_2, a_3, \dots \in \mathbb{C} \text{ and } \exists n \in \mathbb{N} \text{ such that } a_j = 0\ \forall j \geq n\}.$$

In other words, W is the set of all sequences of complex numbers with only finitely many nonzero terms.

(a) Prove that W is a linear subspace of ℓ^2.

(b) Also recall that ℓ^2 is a Hilbert space with inner product

$$\langle (a_1, a_2, a_3, \dots), (b_1, b_2, b_3, \dots) \rangle = \sum_{j=1}^{\infty} a_j \overline{b_j}.$$

Is W a closed linear subspace of ℓ^2? Prove that your answer is correct.

Hint: Consider the following sequence of sequences.

$x_1 = (1, 0, 0, 0, 0 \dots)$

$x_2 = (1, \frac{1}{2}, 0, 0, 0, \dots)$

$x_3 = (1, \frac{1}{2}, \frac{1}{3}, 0, 0, 0, \dots)$

$\vdots$

$x_n = (1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, 0, 0, 0, \dots)$

$\vdots$

SOLUTIONS:

(a) Subset. First observe that $W \subset \ell^2$.

Zero vector. Also $(0, 0, 0, \dots) \in W$.

Closed under addition. Now suppose that $(a_1, a_2, a_3, \dots), (b_1, b_2, b_3, \dots) \in W$. Then $a_j = 0$ for all $j \geq n$ for some natural number n, and $b_j = 0$ for all $j \geq m$ for some natural number m. Let $k = \max\{n, m\}$. Then $a_j + b_j = 0 + 0 = 0$ for all $j \geq k$, so

$$(a_1, a_2, a_3, \dots) + (b_1, b_2, b_3, \dots) = (a_1 + b_1, a_2 + b_2, a_3 + b_3, \dots) \in W.$$

Closed under scalar multiplication. Finally, suppose that $(a_1, a_2, a_3, \dots) \in W$ and $\lambda \in \mathbb{C}$. Then $a_j = 0$ for all $j \geq n$ for some natural number n. So $\lambda a_j = 0$ for all $j \geq n$, so $\lambda(a_1, a_2, a_3, \dots) = (\lambda a_1, \lambda a_2, \lambda a_3, \dots) \in W$.

(b) Observe that $x_j \in W$ for all j.

Let $x = (1, \frac{1}{2}, \frac{1}{3}, \dots \frac{1}{n}, \frac{1}{n+1}, \dots) = (\frac{1}{j})_{j=1}^{\infty}$. Because $x \notin W$, it will suffice to show that $x_m \to x$ as $m \to \infty$.

To do so, it is enough to show that $\|x - x_m\| \to 0$ as $m \to \infty$. (The norm here is the norm induced by the given inner product.)

We compute that

$$x - x_m = (0, 0, \dots, 0, \frac{1}{m+1}, \frac{1}{m+2}, \dots).$$

Therefore $\|x - x_m\|^2 = \langle x - x_m, x - x_m \rangle = \sum_{j=m+1}^{\infty} \frac{1}{j^2}$.

Recall that $\sum_{j=1}^{\infty} \frac{1}{j^2}$ converges. Therefore $\sum_{j=m+1}^{\infty} \frac{1}{j^2} \to 0$ as $m \to \infty$.

Spring 2024 #6. Let M be an arbitrary non-empty set, and define $d : M \times M \to \mathbb{R}$ by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

(a) Show that d defines a metric on M.

(b) Let $\{x_n\}$ be a sequence in M. Show that $\{x_n\}$ converges to x in (M, d) if and only if there exists $N \in \mathbb{N}$ such that $x_n = x$ for $n \geq N$.

(a)

Proof. First observe that $d(x, y)$ is a nonnegative real number for all $x, y \in M$.

Next, $d(x, y) = 0$ if and only if $x = y$, by definition of d.

Next, $d(x, y) = 0 = d(y, x)$ if $x = y$, and $d(x, y) = 1 = d(y, x)$ if $x \neq y$.

Finally, we prove the triangle inequality. That is, let $x, y, z \in M$. We will show that

$$d(x, y) + d(y, z) \geq d(x, z).$$

Case 1: $x = y$

Then $d(x, y) + d(y, z) = 0 + d(x, z) \geq d(x, z)$.

Case 2: $x \neq y$

Then $d(x, y) + d(y, z) = 1 + d(x, z) \geq 1 \geq d(x, z)$. $\square$

(b)

Proof. First suppose $\lim x_n = x$. By def. of limit, this means that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $d(x_n, x) < \epsilon$.

Because we know this "for all" statement is true, we may choose ϵ at will. Let $\epsilon = 1$. Then there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $d(x_n, x) < 1$. But for this metric (the discrete metric), we have that $d(x_n, x) < 1$ if and only if $x_n = x$, so the result follows.

Conversely, suppose that there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $x_n = x$. Let $\epsilon > 0$. It follows that if $n \geq N$, then $d(x_n, x) = d(x, x) = 0 < \epsilon$. Therefore $x_n \to x$. $\square$

Spring 2024 #7. Let $f(t) = t^2$ for $t \in [-\pi, \pi]$, and extend it to be 2π-periodic on $\mathbb{R}$.

(a) Find the Fourier series of $f(t)$ in trigonometric form.

(b) Use the result of Part (a) to show that:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

Solution:

We are given a function $f(t) = t^2$ for $t \in [-\pi, \pi]$, and it is extended to be 2π-periodic on $\mathbb{R}$. We are asked to find the Fourier series of $f(t)$ in trigonometric form and use this to show that:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

(a) Fourier Series of $f(t)$

The Fourier series of a 2π-periodic function $f(t)$ is given by:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nt) + b_n \sin(nt) \right),$$

where the Fourier coefficients a_0, a_n, and b_n are computed as follows:

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\, dt, \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt)\, dt, \quad n \geq 1, \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt)\, dt, \quad n \geq 1. \end{aligned}$$

Step 1: Compute a_0

To find a_0, we use the formula:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} t^2\, dt.$$

Since t^2 is an even function, we can double the integral over $[0, \pi]$:

$$a_0 = \frac{2}{\pi} \int_{0}^{\pi} t^2\, dt.$$

The integral is straightforward:

$$\int_0^{\pi} t^2\, dt = \left[\frac{t^3}{3} \right]_0^{\pi} = \frac{\pi^3}{3}.$$

Thus:

$$a_0 = \frac{2}{\pi} \cdot \frac{\pi^3}{3} = \frac{2\pi^2}{3}.$$

Step 2: Compute a_n

Next, we compute a_n for $n \geq 1$. The formula for a_n is:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} t^2 \cos(nt)\, dt.$$

Since $t^2 \cos(nt)$ is an even function, we can double the integral over $[0, \pi]$:

$$a_n = \frac{2}{\pi} \int_{0}^{\pi} t^2 \cos(nt)\, dt.$$

We can integrate by parts. Let's choose $u = t^2$ and $dv = \cos(nt)\, dt$. Then:

$du = 2t\, dt$,

$v = \frac{\sin(nt)}{n}$.

Using integration by parts:

$$\int_0^{\pi} t^2 \cos(nt)\, dt = \left[\frac{t^2 \sin(nt)}{n} \right]_0^{\pi} - \int_0^{\pi} \frac{2t \sin(nt)}{n}\, dt.$$

The boundary term $\left[\frac{t^2 \sin(nt)}{n} \right]_0^{\pi}$ evaluates to zero because $\sin(n\pi) = 0$ for all integers n. Thus, we are left with:

$$\int_0^{\pi} t^2 \cos(nt)\, dt = -\frac{2}{n} \int_0^{\pi} t \sin(nt)\, dt.$$

Next, we compute the integral $\int_0^{\pi} t \sin(nt)\, dt$ by parts again. Let $u = t$ and $dv = \sin(nt)\, dt$. Then:

$du = dt$, $v = -\frac{\cos(nt)}{n}$.

The integration by parts gives:

$$\int_0^{\pi} t\sin(nt)\,dt = \left[-\frac{t\cos(nt)}{n}\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos(nt)\,dt.$$

The boundary term $\left[-\frac{t\cos(nt)}{n}\right]_0^{\pi}$ evaluates to

$$-\frac{\pi\cos(n\pi)}{n} = (-1)^n\frac{\pi}{n}$$

The remaining integral $\int_0^{\pi}\cos(nt)\,dt$ evaluates to zero for all $n \geq 1$. Thus, we conclude that:

Plugging everything back in, we get

$$a_n = \frac{2}{\pi}\left(-\frac{2}{n}\right)\left((-1)^n\frac{\pi}{n}\right) = \frac{(-1)^n 4}{n^2}.$$

Step 3: Compute b_n

Now, we compute b_n for $n \geq 1$. The formula for b_n is:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2\sin(nt)\,dt.$$

Since $t^2\sin(nt)$ is an odd function, the integral over $[-\pi, \pi]$ is zero:

$$b_n = 0 \quad \text{for all} \quad n \geq 1.$$

Fourier Series

Thus, the Fourier series for $f(t) = t^2$ is:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos(nt) + b_n\sin(nt)) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{(-1)^n 4}{n^2}\cos(nt)$$

(b) Use the Fourier Series to Show the Identity

The function f is differentiable at 0.

Thus the Fourier series for f converges to $f(0)$ when $t = 0$.

Hence we have:

$$f(0) = 0 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{(-1)^n 4}{n^2}.$$

Here we use that $\cos(n \cdot 0) = 0$.

Bringing the sum to the other side and dividing by 4, we find that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$