California State University – Los Angeles

Department of Mathematics

Master's Degree Comprehensive Examination

Analysis Fall 2024

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Do at least two (2) problems from Section 1 below, and at least three (3) problems from Section 2 below. All problems count equally. If you attempt more than two problems from Section 1, the best two will be used. If you attempt more than three problems from Section 2, the best three will be used. Be sure to show your work for all answers.

(1) Write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only.
(3) Begin each problem on a new page.
(4) Assemble the problems you hand in in numerical order.

Exams are graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

SECTION 1 – Do two (2) problems from this section. If you attempt all three, then the best two will be used for your grade.

Spring 2024 #1. Let $\mathbb{R}$ denote the set of real numbers, and let $\mathbb{Q}$ denote the set of rational numbers.

Define $f \colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Prove that for all $a \in \mathbb{R}$, we have that f is not continuous at a.

Spring 2024 #2. Use the definition of limits to show that

$$\lim_{n\to\infty} \frac{1}{(n+1)^2} = 0.$$

Spring 2024 #3. Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in $\mathbb{R}$. Show that

$$\liminf(x_n + y_n) \geq \liminf x_n + \liminf y_n.$$

SECTION 2 – Do three (3) problems from this section. If you attempt more than three, then the best three will be used for your grade.

Spring 2024 #4. Let $C([0,1])$ denote the set of continuous functions on $[0,1]$, with the L^∞ norm defined by

$$\|f\|_{L^\infty} = \sup_{x\in[0,1]} |f(x)|.$$

Let $h \in C([0,1])$. For $f \in L^2([0,1])$, define $T(f) = h(x)f(x)$.

(a) Show that T maps $L^2([0,1])$ to itself.

(b) Show that T is a bounded operator.

(c) Show that $\|T\| \leq \|h\|_{L^\infty}$.

Spring 2024 #5. Recall that

$$\ell^2 = \left\{ (a_1, a_2, a_3, \dots) \mid a_1, a_2, a_3, \dots \in \mathbb{C} \text{ and } \sum_{j=1}^{\infty} |a_j|^2 < \infty \right\}.$$

In other words, ℓ^2 is the set of all square-summable sequences of complex numbers. Let

$$W = \{(a_1, a_2, a_3, \dots) \mid a_1, a_2, a_3, \dots \in \mathbb{C} \text{ and } \exists n \in \mathbb{N} \text{ such that } a_j = 0 \ \forall j \geq n\}.$$

In other words, W is the set of all sequences of complex numbers with only finitely many nonzero terms.

(a) Prove that W is a linear subspace of ℓ^2.

(b) Also recall that ℓ^2 is a Hilbert space with inner product

$$\langle (a_1, a_2, a_3, \dots), (b_1, b_2, b_3, \dots) \rangle = \sum_{j=1}^{\infty} a_j \overline{b_j}.$$

Is W a closed linear subspace of ℓ^2? Prove that your answer is correct.

Hint: Consider the following sequence of sequences.

$x_1 = (1, 0, 0, 0, 0 \dots)$

$x_2 = (1, \frac{1}{2}, 0, 0, 0, \dots)$

$x_3 = (1, \frac{1}{2}, \frac{1}{3}, 0, 0, 0, \dots)$

$\vdots$

$x_n = (1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, 0, 0, 0, \ldots)$

$\vdots$

Spring 2024 #6. Let M be an arbitrary non-empty set, and define $d : M \times M \to \mathbb{R}$ by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

(a) Show that d defines a metric on M.

(b) Let $\{x_n\}$ be a sequence in M. Show that $\{x_n\}$ converges to x in (M, d) if and only if there exists $N \in \mathbb{N}$ such that $x_n = x$ for $n \geq N$.

Spring 2024 #7. Let $f(t) = t^2$ for $t \in [-\pi, \pi]$, and extend it to be 2π-periodic on $\mathbb{R}$.

(a) Find the Fourier series of $f(t)$ in trigonometric form.

(b) Use the result of Part (a) to show that:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$