ALGEBRA COMPREHENSIVE EXAMINATION

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Directions: Answer 5 questions only. You must *answer at least one* from each of linear algebra, groups, and synthesis. Indicate CLEARLY which problems you want us to grade. Otherwise, we will select which ones to grade, and they may not be the ones that you want us to grade. Be sure to show enough work that your answers are adequately supported.

Notation: As usual, $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{Z}_n$, $\mathbb{C}$, and $\mathbb{R}$ denote the sets of natural numbers, rational numbers, integers, integers modulo n, complex numbers, and real numbers respectively, regarded as groups or fields or vector spaces in the usual way.

Linear Algebra

(L1) Let V be a 2-dimensional real vector space and let $T : V \to V$ be a linear transformation. Suppose v_1 and v_2 are vectors in V with $\{v_1, v_2\}$ linearly independent, $T(v_1) = v_2$ and $T(v_2) = v_1 + v_2$. Show that T is invertible.

`Answer:` *Since $\{v_1, v_2\}$ linearly independent in a 2-dimensional space, $\{v_1, v_2\}$ is a basis for V.*

Since T is a linear function from a 2-dimensional space to itself, T is invertible if and only if T is injective (one-to-one), if and only if T is surjective (onto). We will prove both injectivity and surjectivity even though either one of these suffices to prove invertibility.

(a) *T is injective: Suppose that $T(v) = 0$ for some $v \in V$. Then $v = c_1v_1 + c_2v_2$ for some $c_1, c_2 \in \mathbb{R}$, so*

$$\begin{aligned} 0 = T(v) = T(c_1v_1 + c_2v_2) &= c_1T(v_1) + c_2T(v_2) \\ &= c_1v_2 + c_2(v_1 + v_2) = c_2v_1 + (c_1 + c_2)v_2 \end{aligned}$$

Because $\{v_1, v_2\}$ is linearly independent, this implies that $c_2 = c_1 + c_2 = 0$ and hence $c_1 = c_2 = 0$ and $v = 0$. Thus we have shown that $T(v) = 0$ implies $v = 0$, which suffices to show injectivity.

(b) *T is surjective: Let $v \in V$. Then $v = c_1v_1 + c_2v_2$ for some $c_1, c_2 \in \mathbb{R}$. Set $w = (c_2 - c_1)v_1 + c_1v_2$. Then*

$$\begin{aligned} T(w) = (c_2 - c_1)T(v_1) + c_1T(v_2) &= (c_2 - c_1)v_2 + c_1(v_1 + v_2) \\ &= c_1v_1 + c_2v_1 = v \end{aligned}$$

This shows that T is surjective.

(L2) Let V be a vector space, and let $T\colon V \to V$ be a linear operator that is not the identity operator. Suppose T is *idempotent*, meaning that $T^2 = T$. Prove that T is not invertible.

`Answer:` *Suppose towards a contradiction that T is invertible. Then T^{-1} exists, so we can multiply by T^{-1} on both sides of the equation $T^2 = T$:*

$$\begin{aligned} T^{-1}T^2 &= T^{-1}T \\ T^{-1}TT &= T^{-1}T \\ \mathrm{id}_V\, T &= \mathrm{id}_V \\ T &= \mathrm{id}_V\,. \end{aligned}$$

This contradicts the assumption that T is not the identity. Therefore, T is not invertible.

Note: this proof does not assume that V is finite-dimensional. If the student's proof depends on an assumption, explicit or implicit, that V is finite-dimensional, then some small number of points may be deducted.

(L3) Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be non-zero vectors in $\mathbb{R}^n$. Prove that, if $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an orthogonal set, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an independent set.
`Answer:` *Let $c_1, \ldots, c_k \in \mathbb{R}$ such that $c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$. Take the dot product of each side with $\mathbf{v}_i$:*

$$\begin{aligned} 0 = \mathbf{v}_i \cdot \mathbf{0} &= \mathbf{v}_i \cdot (c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k) \\ &= c_1(\mathbf{v}_i \cdot \mathbf{v}_1) + c_2(\mathbf{v}_i \cdot \mathbf{v}_2) + \cdots + c_i(\mathbf{v}_i \cdot \mathbf{v}_i) + \cdots + c_k(\mathbf{v}_i \cdot \mathbf{v}_k). \end{aligned}$$

Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an orthogonal set, we have $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ if $j \neq i$. So each term $c_j(\mathbf{v}_i \cdot \mathbf{v}_j)$ of the sum above is 0 except $c_i(\mathbf{v}_i \cdot \mathbf{v}_i)$. Thus we obtain

$$c_i(\mathbf{v}_i \cdot \mathbf{v}_i) = 0,$$

which implies $c_i = 0$ since $\mathbf{v}_i \neq \mathbf{0}$.

Therefore, if $c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$, then $c_1 = \cdots = c_k = 0$. Therefore, $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an independent set.

Groups

(G1) Let G be a group and let $\phi : S_4 \to G$ be a homomorphism with $(1\,2\,3\,4) \in \ker\phi$. Show that $\ker\phi = S_4$.
`Answer:` *$\ker\phi$ is a normal subgroup of S_4. The only normal subgroups of S_4 are $\{1\}$, $V = \{1, (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}$, A_4 and S_4. The element $(1\,2\,3\,4)$ is clearly not in the first two of these normal subgroups, and it is not in A_4 since $(1\,2\,3\,4)$ is odd. So $\ker\phi = S_4$.*

OR

$\ker\phi$ is a normal subgroup of S_4. A normal subgroup is a union of conjugacy classes. There are six 4-cycles in the conjugacy class of $(1\,2\,3\,4)$ so these six 4-cycles are in $\ker\phi$. In addition, $(1\,2\,3\,4)(1\,2\,4\,3) = (1\,3\,2)$, so this 3-cycle and its 7 conjugates are in $\ker\phi$. So far we have identified 14 elements in $\ker\phi$. But, since the order of $\ker\phi$ must divide $|S_4| = 24$, this can happen only if $\ker\phi = S_4$.

(G2) Let G be finite cyclic group of order m and let H be finite cyclic group of order n. Prove that $G \times H$ is cyclic if and only if $\gcd(m, n) = 1$.

`Answer:` *Suppose* $\gcd(m,n) = 1$. *Let* $g \in G$ *and* $h \in H$ *be such that* $G = \langle g \rangle$ *and* $H = \langle h \rangle$. *Note that the order of* (g,h) *in* $G \times H$ *is* $|(g,h)| = \text{lcm}(|g|,|h|) = \text{lcm}(m,n)$. *But* $\gcd(m,n) = 1 \Rightarrow \text{lcm}(m,n) = mn$. *Thus* $|(g,h)| = |G||H| = |G \times H|$, *and* (g,h) *is a generator for* $G \times H$.

Conversely, suppose $G \times H$ *is cyclic and let* (a,b) *be a generator for* $G \times H$. *Then* $|(a,b)| = |G \times H| = |G||H| = mn$. *On the other hand,* $|(a,b)| = \text{lcm}(|a|,|b|)$. *Note that* a *is a generator for* G*:*
If $g \in G$, *then* $(g, e_H) \in G \times H$. *Therefore, there exists* k *such that* $(a,b)^k = (a^k, b^k) = (g, e_H)$. *So* $g = a^k$ *and* $G = \langle a \rangle$ *because* g *is arbitrary. Analogously,* $H = \langle b \rangle$. *Then,* $mn = |(a,b)| = \text{lcm}(|a|,|b|) = \text{lcm}(m,n)$. *Thus* $\gcd(m,n) = 1$.

(G3) Prove that $(\mathbb{Q}, +)$ is not cyclic.

`Answer:` *Let* $a \in \mathbb{Q}$ *and consider the cyclic subgroup* $\langle a \rangle$. *If* $a = 0$, *then* $\langle a \rangle = \{0\}$ *and so* $\langle a \rangle \neq \mathbb{Q}$ *in this trivial case. Otherwise, if* $a \neq 0$, *then* $\langle a \rangle = \{na \mid n \in \mathbb{Z}\}$. *In particular,* $a/2 \in \mathbb{Q}$ *is not in* $\langle a \rangle$. *(In detail, if* $a/2 \in \langle a \rangle$, *then* $a/2 = na$ *for some* $n \in \mathbb{Z}$. *Since* $a \neq 0$, *we can cancel to get* $1/2 = n \in \mathbb{Z}$, *a contradiction.) Thus* $\langle a \rangle \neq \mathbb{Q}$ *in this case as well. Since no cyclic subgroup of* $\mathbb{Q}$ *is equal to* $\mathbb{Q}$, $\mathbb{Q}$ *is not cyclic.*

Synthesis

(S1) Let $M_{n\times n}(\mathbb{R})$ be set of $n \times n$ matrices with entries in $\mathbb{R}$ and let $GL_n(\mathbb{R})$ be the group of invertible $n \times n$ matrices with matrix multiplication as group operation. Recall that $A \in M_{n\times n}(\mathbb{R})$ is orthogonal if $A^T A = AA^T = I$, where I is the identity.
Let $O_n(\mathbb{R}) = \{A \in M_{n\times n}(\mathbb{R}) : A \text{ is orthogonal } \}$. Prove that $O_n(\mathbb{R})$ is a subgroup of $GL_n(\mathbb{R})$.

`Answer:` *The identity* I *is obviously orthogonal. Also, if* $A, B \in O_n(\mathbb{R})$ *then* $(AB)^T AB = B^T A^T AB = B^T IB = B^T B = I$. *Similarly* $AB(AB)^T = I$ *so,* AB *is orthogonal. Finally, it's clear that if* $A \in O_n(\mathbb{R})$ *then* $A^{-1} = A^T$. *Moreover,* $A^T (A^T)^T = A^T A = I$. *And* $(A^T)^T A^T = AA^T = I$. *So,* $A^{-1} \in O_n(\mathbb{R})$.

(S2) Find the center of $GL_2(\mathbb{R})$. Explain.
Reminder: The **center** of a group G is $Z(G) = \{g \in G \mid gh = hg \text{ for all } h \in G\}$.

`Answer:` *Suppose* $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *is in the center of* $GL_2(\mathbb{R})$. *Then this matrix commutes with all other matrices, in particular,*

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

The first of these matrix equations holds if and only if $c = a - d = 0$ *and the second holds if and only if* $b = a - d = 0$. *Thus any matrix in the center has the form* $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = aI$ *for some* $a \in \mathbb{R}$. *Since any scalar multiple of* I *commutes with all other matrices, and so is in the center, we have shown that* $Z(GL_2(\mathbb{R})) = \{aI \mid a \in \mathbb{R}^*\}$.

(S3) Define $H = \{A \in GL_2(\mathbb{R})\colon \det(A) \in \mathbb{Q}\}$ (where $\mathbb{R}$ denotes the set of real numbers and $\mathbb{Q}$ denotes the set of rational numbers).

(a) Prove that H is a subgroup of $GL_2(\mathbb{R})$.

(b) Show that H is not equal to $GL_2(\mathbb{Q})$.

`Answer:` *First of all,* $\det(I) = 1 \in \mathbb{Q}$*, so* $I \in H$*.*

If $A, B \in H$*, then* $\det(A), \det(B) \in \mathbb{Q}$*, so* $\det(AB) = \det(A)\det(B) \in \mathbb{Q}$ *(since the product of two rational numbers is rational). Thus* $AB \in H$*.*

If $A \in H$*, then* $\det(A) \in \mathbb{Q}$*, so* $\det(A^{-1}) = \frac{1}{\det(A)} \in \mathbb{Q}$ *(since the reciprocal of a rational number is rational). Therefore,* H *is a subgroup.*

Furthermore, $H \neq GL_2(\mathbb{Q})$*: indeed,* $\begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}$ *is in* H *since its determinant is* 2*, but it is not in* $GL_2(\mathbb{Q})$ *since* $\sqrt{2} \notin \mathbb{Q}$*.*