ALGEBRA COMPREHENSIVE EXAMINATION

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<u>Directions</u>: Answer 5 questions only. You must *answer at least one* from each of linear algebra, groups, and synthesis. Indicate CLEARLY which problems you want us to grade. Otherwise, we will select which ones to grade, and they may not be the ones that you want us to grade. Be sure to show enough work that your answers are adequately supported.

Linear Algebra

(L1) Let V be a 2-dimensional real vector space and let $T : V \to V$ be a linear transformation. Suppose v_1 and v_2 are vectors in V with $\{v_1, v_2\}$ linearly independent, $T(v_1) = v_2$ and $T(v_2) = v_1 + v_2$. Show that T is invertible.

(L2) Let V be a vector space, and let $T\colon V \to V$ be a linear operator that is not the identity operator. Suppose T is *idempotent*, meaning that $T^2 = T$. Prove that T is not invertible.

(L3) Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be non-zero vectors in $\mathbb{R}^n$. Prove that, if $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an orthogonal set, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an independent set.

Groups

(G1) Let G be a group and let $\phi : S_4 \to G$ be a homomorphism with $(1\,2\,3\,4) \in \ker \phi$. Show that $\ker \phi = S_4$.

(G2) Let G be finite cyclic group of order m and let H be finite cyclic group of order n. Prove that $G \times H$ is cyclic if and only if $\gcd(m, n) = 1$.

(G3) Prove that $(\mathbb{Q}, +)$ is not cyclic.

Synthesis

(S1) Let $M_{n\times n}(\mathbb{R})$ be set of $n \times n$ matrices with entries in $\mathbb{R}$ and let $GL_n(\mathbb{R})$ be the group of invertible $n \times n$ matrices with matrix multiplication as group operation. Recall that $A \in M_{n\times n}(\mathbb{R})$ is orthogonal if $A^T A = AA^T = I$, where I is the identity.
Let $O_n(\mathbb{R}) = \{A \in M_{n\times n}(\mathbb{R}) : A \text{ is orthogonal }\}$. Prove that $O_n(\mathbb{R})$ is a subgroup of $GL_n(\mathbb{R})$.

(S2) Find the center of $GL_2(\mathbb{R})$. Explain.
Reminder: The **center** of a group G is $Z(G) = \{g \in G \mid gh = hg \text{ for all } h \in G\}$.

(S3) Define $H = \{A \in GL_2(\mathbb{R})\colon \det(A) \in \mathbb{Q}\}$ (where $\mathbb{R}$ denotes the set of real numbers and $\mathbb{Q}$ denotes the set of rational numbers).
(a) Prove that H is a subgroup of $GL_2(\mathbb{R})$.
(b) Show that H is not equal to $GL_2(\mathbb{Q})$.