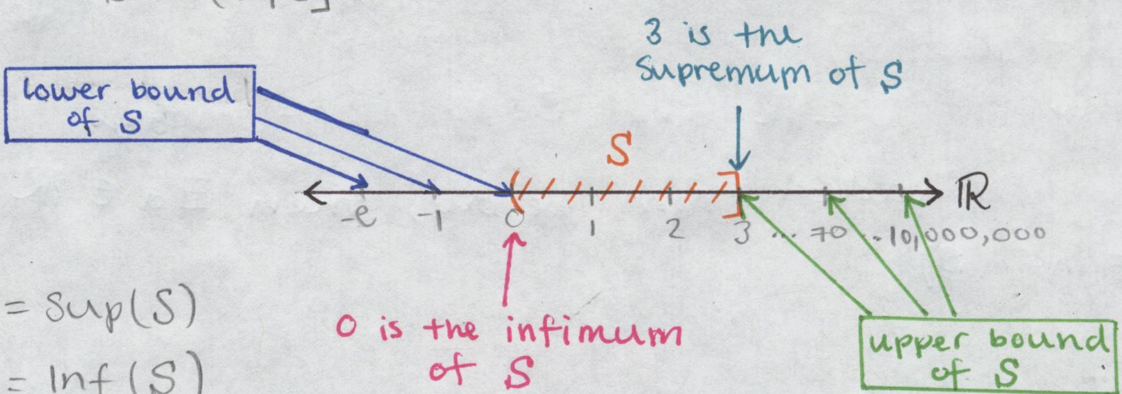
The Completeness Axiom

HW 1

Def: Let $S \subseteq \mathbb{R}$, where $S \neq \emptyset$.

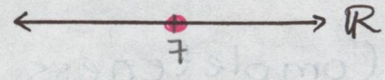
- We say that S is **bounded from above** if there exists $b \in \mathbb{R}$ where $x \leq b$ for all $x \in S$. If this is the case then we call b an **upper bound** for S.
  Furthermore, if b is an upper bound for S and $b \leq c$ for all upper bounds of S, then we call b the **least upper bound** or **supremum** of S, and we write $b = \sup(S)$.
- We say that S is **bounded from below** if there exists $a \in \mathbb{R}$ where $a \leq x$ for all $x \in S$. If this is the case we call a a **lower bound** for S.
  Furthermore, if a is a lower bound for S and $c \leq a$ for all lower bounds of S, then we call a the **greatest lower bound** or **infimum** of S, and we write $a = \inf(S)$.

Example: $S = (0, 3]$



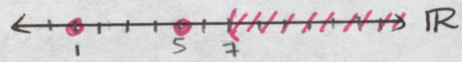
$3 = \sup(S)$
$0 = \inf(S)$.

Example: $S = \{7\}$



$7 = \sup(S) = \inf(S)$

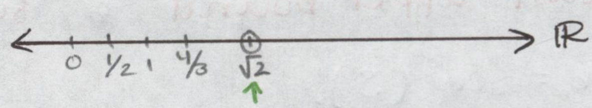
Example: $S = \{1, 5\} \cup (7, \infty)$



- $\sup(S)$ does not exist.
- $\inf(S) = 1$

Consider the subset of $\mathbb{Q}$ defined by

$$S = \{x \in \mathbb{Q} \mid x < \sqrt{2}\}$$



there are infinitely many rationals in here.

- $\sqrt{2}$ is the least upper bound for S but $\sqrt{2} \notin \mathbb{Q}$
- $\mathbb{Q}$ has "holes" in it.

$\mathbb{R}$ fills in the "holes" in $\mathbb{Q}$ by adding the irrational numbers
We will assume the following property of $\mathbb{R}$

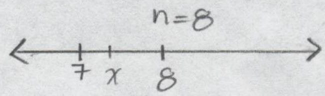
The Completeness Axiom

- Let S be a non-empty subset of $\mathbb{R}$.
  If S is bounded from above, then $\exists\, b \in \mathbb{R}$ where $b = \sup(S)$
- Similarly, if $S \subseteq \mathbb{R}$ and S is non-empty and S is bounded from below, then $\exists\, a \in \mathbb{R}$ with $a = \inf(S)$

Theorem: (The Archimedean Property)

If $x \in \mathbb{R}$, then $\exists\, n \in \mathbb{N}$ with $x < n$

Ex: $x = 7.3142\ldots$

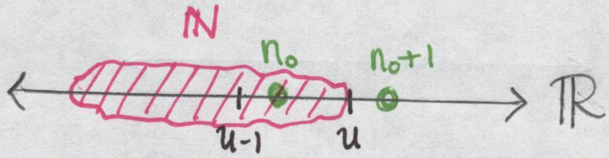


Proof: Suppose this isn't true

then, $\exists\, x \in \mathbb{R}$ where $n \leq x \ \forall n \in \mathbb{N}$.

then x is an upper bound for $\mathbb{N}$.

By the completeness axiom, since $\mathbb{N}$ is a non-empty subset of $\mathbb{R}$ and $\mathbb{N}$ has an upper bound, we have that $u = \sup(\mathbb{N})$ exists



Since u is the least upper bound of $\mathbb{N}$, then $u-1$ is not an upper bound for $\mathbb{N}$. So $\exists\, n_0 \in \mathbb{N}$ where $u-1 \leq n_0$ so $u \leq n_0 + 1$

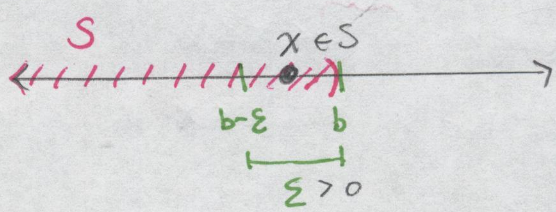
But $n_0 + 1 \in \mathbb{N}$ and $u = \sup(\mathbb{N})$

Contradiction ▨

Theorem: (The Useful inf/sup fact)

Let $S \subseteq \mathbb{R}$ be non-empty

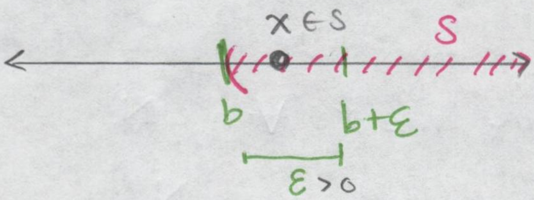
(a) Suppose S is bounded from above by some $b \in \mathbb{R}$. Then b is the supremum of S iff for every $\varepsilon > 0\ \exists\, x \in S$ with $b - \varepsilon < x \leq b$



(b) Suppose S is bounded from below by $b \in \mathbb{R}$

Then b is the infimum of S iff for every $\varepsilon > 0$

$\exists\, x \in S$ with $b \leq x < b + \varepsilon$.

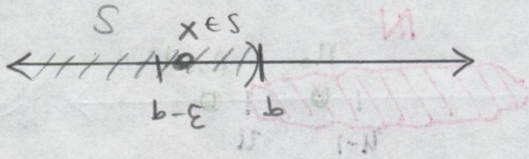


Proof: we prove (a)

let $b \in \mathbb{R}$ be an upper bound for S

($\Rightarrow$) Assume $b = \sup(S)$

Let $\varepsilon > 0$



Since $b - \varepsilon < b$ and b is the least upper bound of S, then $b - \varepsilon$ is not an upper bound of S

So $\exists\, x \in S$ with $b - \varepsilon < x$.

Since $x \in S$ and $b = \sup(S)$, we have $x \leq b$.

so $b - \varepsilon < x \leq b$

($\Leftarrow$) Let b be an upper bound for S such that for every $\varepsilon > 0$ $\exists\, x \in S$ with $b - \varepsilon < x \leq b$. we want to show that b is the supremum of S, or the least upper bound of S.

Suppose that $c \in \mathbb{R}$ is an upper bound for S.

we must show that $b \leq c$

well what would happen if $c < b$

let $\varepsilon = b - c > 0$.

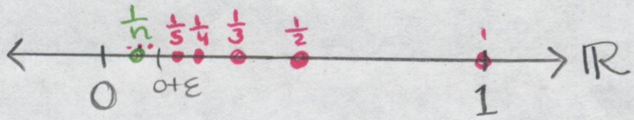
By assumption $\exists\, x \in S$ with $\overbrace{b-\varepsilon}^{c} < x \leq b$

But c is an upper bound for S, so $c < x$ is a contradiction.

Thus $b \leq c$ ▨

Ex: Let $X = \{\frac{1}{n} \mid n \in \mathbb{N}\}$

what is $\inf(X)$ if it exists and prove it.



Claim: $0 = \inf(X)$

- $0 < \frac{1}{n} \;\forall n \in \mathbb{N}$. So 0 is a lower bound for X
- Let $\varepsilon > 0$. we need to find $x \in X$ where $0 \le x < 0 + \varepsilon$.

since $\frac{1}{\varepsilon} \in \mathbb{R}$ $\exists n \in \mathbb{N}$ with $n > \frac{1}{\varepsilon}$ (Archimedean Property)

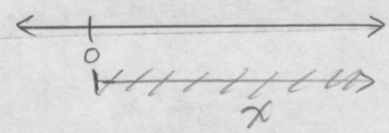
Then $\frac{1}{n} < \varepsilon$, so $x = \frac{1}{n} \in X$ and $0 < \frac{1}{n} < 0 + \varepsilon$ $\boxtimes$

By the useful inf/sup fact $0 = \inf(X)$

HW 1 #2

let $x \in \mathbb{R}$ with $x \ge 0$

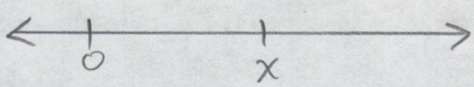
If $x < \varepsilon \;\forall \varepsilon > 0$, then $x = 0$



Proof: Assume $x \ge 0$ and $x < \varepsilon \;\forall \varepsilon > 0$

we know either $x = 0$ or $x > 0$.

let's rule out the $x > 0$ case



Suppose $\underline{x > 0}$ then if we divide by 2 $\frac{x}{2} > 0$. set $\varepsilon = \frac{x}{2}$

By assumption $x < \varepsilon = \frac{x}{2}$

Then $x < \frac{x}{2}$

So $\frac{x}{2} < 0$ So $\underline{x < 0}$ $\Rightarrow\Leftarrow$ contradiction!

Therefore $x = 0$

$\boxtimes$